

Global fluctuations for 1D log-gas dynamics

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We study in this article the hydrodynamic limit in the macroscopic regime of the coupled system of stochastic differential equations,

$$d\lambda_t^i = \frac{1}{\sqrt{N}} dW_t^i - V'(\lambda_t^i)dt + \frac{\beta}{2N} \sum_{j \neq i} \frac{dt}{\lambda_t^i - \lambda_t^j}, \quad i = 1, \dots, N, \quad (0.1)$$

with $\beta > 1$, sometimes called *generalized Dyson's Brownian motion*, describing the dissipative dynamics of a log-gas of N equal charges with equilibrium measure corresponding to a β -ensemble, with sufficiently regular convex potential V . The limit $N \rightarrow \infty$ is known to satisfy a mean-field Mac-Kean-Vlasov equation. We prove that, for suitable initial conditions, fluctuations around the limit are Gaussian and satisfy an explicit PDE.

The proof is very much indebted to the harmonic potential case treated in Israelsson [13]. Our key argument consists in showing that the time-evolution generator may be written in the form of a transport operator on the upper half-plane, plus a bounded non-local operator interpreted in terms of a signed jump process. As an essential technical argument ensuring the convergence of the above scheme, we give an N -independent large-deviation type estimate for the probability that $\sup_{t \in [0, T]} \max_{i=1, \dots, N} |\lambda_t^i|$ is large, based on a multi-scale argument and entropic bounds.

Keywords: random matrices, Dyson's Brownian motion, log-gas, beta-ensembles, hydrodynamic limit, Stieltjes transform, entropy, signed jump process

Mathematics Subject Classification (2010): 60B20; 60F05; 60G20; 60J60; 60J75; 60K35

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1 Introduction and statement of main results

1.1 Introduction

Let $\beta \geq 1$ be a fixed parameter, and $N \geq 1$ an integer. We consider the following system of coupled stochastic differential equations driven by N independent standard Brownian motions $(W_t^1, \dots, W_t^N)_{t \geq 0}$,

$$d\lambda_t^i = \frac{1}{\sqrt{N}} dW_t^i - V'(\lambda_t^i) dt + \frac{\beta}{2N} \sum_{j \neq i} \frac{dt}{\lambda_t^i - \lambda_t^j}, \quad i = 1, \dots, n \quad (1.1)$$

Letting

$$\mathcal{W}(\{\lambda^i\}_i) := \sum_{i=1}^N V(\lambda^i) - \frac{\beta}{4N} \sum_{i \neq j} \log(\lambda^i - \lambda^j), \quad (1.2)$$

we can rewrite (1.1) as $d\lambda_t^i = \frac{1}{\sqrt{N}}dW_t^i - \nabla_i \mathcal{W}(\lambda_t^1, \dots, \lambda_t^N)dt$. Thus the corresponding equilibrium measure,

$$d\mu_{eq}^N(\{\lambda^i\}_i) = \frac{1}{Z_V^N} e^{-2N\mathcal{W}(\{\lambda^i\}_i)} = \frac{1}{Z_V^N} \left(\prod_{j \neq i} |\lambda^j - \lambda^i| \right)^{\beta/2} \exp \left(-2N \sum_{i=1}^N V(\lambda^i) \right) d\lambda^1 \dots d\lambda^N \quad (1.3)$$

is that of a β -log gas with confining potential V .

Let us start with a historical overview of the subject as a motivation for our study. This system of equations was originally considered in a particular case by Dyson [6] who wanted to describe the Markov evolution of a Hermitian matrix M_t with i.i.d. increments dG_t taken from the Gaussian unitary ensemble (GUE). In Dyson's idea, this matrix-valued process was to be a matrix analogue of Brownian motion. The latter time-evolution being invariant through conjugation by unitary matrices, we may project it onto a time-evolution of the set of eigenvalues $\{\lambda_t^1, \dots, \lambda_t^N\}$ of the matrix, and obtain (1.1) with $\beta = 2$ and $V \equiv 0$. Keeping $\beta = 2$, it is easy to prove that (1.1) is equivalent to a generalized matrix Markov evolution, $dM_t = dG_t - V'(M_t)dt$. The Gibbs measure

$$\mathcal{P}_V^N(M) = \frac{1}{Z_N} e^{-N\text{Tr}V(M)} dM, \quad dM = \prod_{i=1}^N dM_{ii} \prod_{1 \leq i < j \leq n} d\text{Re } M_{ij} d\text{Im } M_{ij}$$

can then be proved to be an equilibrium measure. Such measures, together with their projection onto the eigenvalue set, $\mu_{eq}^N(\{\lambda^1, \dots, \lambda^N\})$, are the main object of random matrix theory, see e.g. [20], [1], [24]. The *equilibrium eigenvalue distribution* can be studied by various means, in particular using orthogonal polynomials with respect to the weight $e^{-NV(\lambda)}$. The scaling in N (called *macroscopic scaling* in random matrix theory) ensures the convergence of the random point measure $X^N := \frac{1}{N} \sum_{i=1}^N \delta_{\lambda^i}$ to a deterministic measure μ_V with *compact* support and density ρ when $N \rightarrow \infty$ (see e.g. [14], Theorem 2.1). One finds e.g. the well-known semi-circle law, $\rho(x) = \frac{1}{\pi} \sqrt{2 - x^2}$, when $V(x) = x^2/2$. Looking more closely at the limit of the point measure, one finds for arbitrary *polynomial* V (Johansson [14]) Gaussian fluctuations of order $O(1/N)$, contrasting with the $O(1/\sqrt{N})$ scaling of fluctuations for the means of N independent random variables, typical of the central limit theorem. Assuming that the support of the measure is connected (this essential "one-cut" condition holding in particular for V *convex*), Johansson proves that the *covariance* of the limiting law depends on V only through the support of the measure – it is thus *universal* up to a scaling coefficient –, while the means is equal to ρ , plus an apparently non-universal correction in $O(1/N)$.

Then Rogers and Shi [26], disregarding the random matrix background, studied directly for its sake the system (1.1) in the case where V is harmonic (i.e. quadratic) and $\beta = 2$, which we call *Hermite case* henceforth (by reference to the corresponding class of equilibrium orthogonal polynomials), proving in particular the following two facts:

- (i) two arbitrary eigenvalues never collide, which implies the non-explosion of (1.1);
- (ii) the random point process $X_t^N := \frac{1}{N} \sum_{i=1}^N \delta_{\lambda_t^i}$ satisfies in the limit $N \rightarrow \infty$ a deterministic hydrodynamic equation of Mc-Kean Vlasov type, namely, the asymptotic

density

$$\rho_t \equiv X_t := \text{w-lim}_{N \rightarrow \infty} X_t^N \quad (1.4)$$

satisfies the PDE

$$\frac{\partial \rho_t(x)}{\partial t} = \frac{\partial}{\partial x} \left(\left(x - p.v. \int \frac{dy}{x-y} \rho_t(y) \right) \rho_t(x) \right), \quad (1.5)$$

where $p.v. \int \frac{dy}{x-y} \rho_t(y)$ is a principal value integral.

In the case studied by Rogers and Shi, explicit formulas for finite N are known for the Markov generator in the form of a determinant, called *extended kernel*, see e.g. [8], chapter XI, or [9]), whose asymptotics for $N \rightarrow \infty$ may in principle be used to study the macroscopic limit. This is accomplished by noting that a simple conjugation trick turns the generator of the process into an N -particle Hamiltonian with a *one-body* potential only, whose eigenfunctions are deduced from those of one-particle Hamiltonians (actually, harmonic oscillators). For general V , however, in marked contrast with respect to the equilibrium case, no explicit formulas are known for finite N , even for $\beta = 2$, since the conjugation trick produces a supplementary two-body potential making the spectral problem unsolvable. Then for $\beta \neq 2$, the finite N equilibrium measure is not fully understood, even in the harmonic case, see [31].

This makes the direct study of (1.1) for general V and β all the more interesting. Whereas the PDE appearing in the hydrodynamical limit is known [18], the law of fluctuations is not known in general, and it is the purpose of this study, and of the forthcoming article [30], to fill this gap. S. Li, X.-D. Li and Y.-X. Xie [18], generalizing properties (i) and (ii) above, prove that the random point process X_t^N satisfies in the limit $N \rightarrow \infty$ a generalization of the above Mc-Kean Vlasov equation, namely,

$$\frac{\partial \rho_t(x)}{\partial t} = \frac{\partial}{\partial x} \left(\left(V'(x) - \frac{\beta}{2} p.v. \int \frac{dy}{x-y} \rho_t(y) \right) \rho_t(x) \right) \quad (1.6)$$

The equilibrium measure ρ , defined as the solution of the integral equation $\frac{\beta}{2} p.v. \int \frac{dy}{x-y} \rho(y) = V'(x)$, cancels the right-hand side of (1.6). Replacing in (1.1) $\frac{\beta}{2N} \sum_{j \neq i} \frac{1}{\lambda_t^i - \lambda_t^j}$ with

$-\frac{1}{N} \sum_{j \neq i} \nabla U(\lambda_t^i - \lambda_t^j)$ where U is some convex two-body potential satisfying some very general properties of regularity and growth at infinity, one may show that there appears in the same limit an equation similar to (1.6),

$$\frac{\partial \rho_t(x)}{\partial t} = \frac{\partial}{\partial x} \left(\left(V'(x) + \frac{\beta}{2} \int dy U'(x-y) \rho_t(y) \right) \rho_t(x) \right) \quad (1.7)$$

Solutions of this type of equations, common in plasma theory and the study of granular media [12, 3] and in particular, the rate of convergence of these to equilibrium, have been studied in details using Otto's infinite dimensional differential calculus [22] in a series of papers, see e.g. [4], [23], [32]. However, as already noted by S. Li, X.-D. Li and Y.-X. Xie, the range of applicability of these papers, written under the assumption that U be *Lipschitz*, does not seem to extend to our case when $U(x) = -c \log |x|$. Since formally the law of fluctuations is obtained by linearizing the system of equations (1.1) around its macroscopic limit ρ , it is clear that one must find some way to deal with (1.6).

Rogers' and Shi's approach to (1.1) has been successfully generalized to the case of a harmonic potential with arbitrary β by Israelsson [13] and Bender [2]. The present study – and our article in preparation, see below – owes very much to these two articles, so let us describe to some extent their contents. There are two main ideas. Let $Y_t^N := N(X_t^N - X_t)$ be the rescaled fluctuation process for finite N ; we want to prove that $Y_t^N \xrightarrow{\text{law}} Y_t$ when $N \rightarrow \infty$ and identify the law of the process $(Y_t)_{t \geq 0}$. First, Itô's formula implies that

$$d\langle Y_t^N, f_t \rangle = \frac{1}{2} \left(1 - \frac{\beta}{2}\right) \langle X_t^N, f_t'' \rangle dt + \frac{1}{\sqrt{N}} \sum_{i=1}^N f_t'(\lambda_t^i) dW_t^i \quad (1.8)$$

if the test functions $(f_t)_{0 \leq t \leq T}$, $f_t : \mathbb{R} \rightarrow \mathbb{R}$ solve the following linear PDE

$$\frac{\partial f_t}{\partial t}(x) = V'(x)f_t'(x) - \frac{\beta}{4} \int \frac{f_t'(x) - f_t'(y)}{x - y} (X_t^N(dy) + X_t(dy)) \quad (1.9)$$

(see Proposition 1.3 below). Eq. (1.9) is a dualized, linearized version of (1.6). Second, eq. (1.9) may be integrated in the harmonic case by means of a *Stieltjes transform* (see Definition 1.2). Namely, the family of functions $\{\frac{c}{\cdot - z}\}_{c \in \mathbb{C}, z \in \mathbb{C} \setminus \mathbb{R}}$ is preserved by (1.9). The solution $\frac{c_t}{\cdot - z_t}$ at time t with terminal condition $\frac{c_T}{\cdot - z_T} = \frac{c}{\cdot - z}$ is obtained by solving two coupled ordinary differential equations for c_t and z_t depending on X and the random point measure X^N (see [13], Lemma 2). Bender interprets these equations as *characteristic equations* for a generalized transport operator (see section 6) which is never stated explicitly. Then (at least formally), Itô's formula (see [13], p. 29) makes it possible to find explicitly the Markov kernel in the limit $N \rightarrow \infty$. Namely, consider a finite number of points $(z^k)_k$ in $\mathbb{C} \setminus \mathbb{R}$, and the solutions $(z_t^k)_{t \leq T}$ of the corresponding characteristic equations with terminal condition $(z_T^k)_k$. Letting $f_t(x) := \sum_k \frac{c_t^k}{x - z_t^k}$ and $\phi_{f_t}(Y_t^N) := e^{i\langle Y_t^N, f_t \rangle}$,

$$\mathbb{E}[\phi_{f_T}(Y_T) | \mathcal{F}_t] = \mathbb{E}[\phi_{f_t}(Y_t)] \exp \left(\frac{1}{2} \int_t^T \left[i \left(1 - \frac{\beta}{2}\right) \langle X_s, f_s'' \rangle - \langle X_s, (f_s')^2 \rangle \right] ds \right) \quad (1.10)$$

Since functions f of the above form are dense in some appropriate Sobolev space, formula (1.10) allows to conclude that the limit process is Gaussian. Then Bender solves explicitly the characteristic equations, which take on a particularly simple form in the harmonic case, and deduces first the covariance of the Stieltjes transform of the fluctuation process, $\text{Cov}(U_{t_1}(z_1), U_{t_2}(z_2))$, $U_t(z) := \langle Y_t, \frac{1}{\cdot - z} \rangle$, and then (taking boundary values and using the Plemelj formula, see section 7), the (distribution-valued) covariance kernel $\text{Cov}(Y_{t_1}(x_1), Y_{t_2}(x_2))$.

Our approach for the case of a general potential has exactly the same starting point, but dealing with eq. (1.9) turns out to be more complicated than in the harmonic case. The reason is that the family of functions $\left\{ \frac{c}{\cdot - z} \right\}_{c \in \mathbb{C}, z \in \mathbb{C} \setminus \mathbb{R}}$ is no more preserved by (1.9): this is easily seen if V is a polynomial or extends analytically to a strip around the real axis, since

$$-V'(x) \partial_x \left(\frac{1}{x - z} \right) = (V'(z) \partial_z + V''(z)) \left(\frac{1}{x - z} \right) + \left(\frac{V^{(3)}(z)}{2!} + \frac{V^{(4)}(z)}{3!} (x - z) + \dots \right) \quad (1.11)$$

features extra unwanted polynomial terms. In practice we need only assume that V is sufficiently regular, and (letting $a := \operatorname{Re} z$) write instead, for a in a neighbourhood of the support of the random point measure

$$V'(x) = V'(a) + V''(a)(x-a) + V'''(a)\frac{(x-a)^2}{2} + (x-a)^3 W_a(x-a), \quad (1.12)$$

and find for the first three terms,

$$-V'(a)\partial_x \left(\frac{1}{x-z} \right) = V'(a)\partial_a \left(\frac{1}{x-z} \right), \quad -V''(a)(x-a)\partial_x \left(\frac{1}{x-z} \right) = V''(a)(1+b\partial_b) \frac{1}{x-z}, \quad (1.13)$$

$$-V'''(a)\frac{(x-a)^2}{2}\partial_x \left(\frac{1}{x-z} \right) = \frac{1}{2}V'''(a) + \frac{1}{2}V'''(a)(2ib+b^2\partial_a) \frac{1}{x-z}, \quad (1.14)$$

defining a generalized transport operator $-V'(a)\partial_a - V''(a)(1+b\partial_b) - \frac{1}{2}V'''(a)(2ib+b^2\partial_a)$. The new piece is the last (Taylor's remainder) term in (1.12). We must give up at this point the idea that the time-evolution is a simple characteristic evolution, and prove that the Taylor remainder produces instead a *non-local kernel*. The main tool here is the use of *Stieltjes decompositions* of order κ (see Definition 2.3): for any $b_{max} > 0$ and $\kappa = 0, 1, 2, \dots$, any sufficient regular, integrable function $f : \mathbb{R} \rightarrow \mathbb{R}$ may be written as an integral over the strip $\Pi_{b_{max}} := \{a \pm ib \mid 0 < |b| < b_{max}\}$

$$f(x) = \int_{-\infty}^{+\infty} da \int_{-b_{max}}^{b_{max}} db \frac{|b|^{1+\kappa}}{(1+\kappa)!} f_z(x) h(a, b). \quad (1.15)$$

Explicit Stieltjes decompositions are produced in [13], Lemma 9. However, Stieltjes decompositions are not unique; part of the job consists in *choosing* Stieltjes decompositions with good properties. Let $\kappa' \geq \kappa \geq 0$. Inserting the time-evolution operator (1.9) into (1.15), we prove that the *Taylor remainder term* (see §3.8), to which one must add an inessential off-support contribution (see §3.3) and boundary terms (see §3.9), may be realized as a non-local operator $|b|^{\kappa'-\kappa} \mathcal{H}_{nonlocal}^{\kappa';\kappa}(t)$ acting on the coefficient function h ,

$$|b|^{\kappa'-\kappa} (\mathcal{H}_{nonlocal}^{\kappa';\kappa}(t))(h)(a, b) := |b|^{\kappa'-\kappa} \int_{-\infty}^{+\infty} da_T \int_{-b_{max}}^{b_{max}} db_T g_{nonlocal}^{\kappa';\kappa}(a, b; a_T, b_T) h(a_T, b_T) \quad (1.16)$$

(see (3.42) and (3.79)) for some kernel $g_{nonlocal}^{\kappa';\kappa}(a, b; a_T, b_T)$ such that

$$\mathcal{H}_{nonlocal}^{\kappa}(t) \equiv \mathcal{H}_{nonlocal}^{\kappa;\kappa} : L^1(\Pi_{b_{max}}) \rightarrow L^1(\Pi_{b_{max}}) \quad (1.17)$$

and

$$\mathcal{H}_{nonlocal}^{\kappa+1;\kappa}(t) : L^1(\Pi_{b_{max}}) \rightarrow L^1(\Pi_{b_{max}}) \quad (1.18)$$

are *bounded*. From (1.18) we get: $b\mathcal{H}_{nonlocal}^{\kappa+1;\kappa}(h)(a, b) = \tilde{h}(a, b)$ with $\tilde{h} \in L^1(\Pi_{b_{max}}, |b|^{-1} da db)$. In other words, the nonlocal part of the time evolution is (in some weak sense) *regularizing* near the real axis.

The sum of the *other contributions* due to (i) the $(\frac{1}{x})$ -potential; (ii) *the first three terms* (constant, linear and quadratic terms) in (1.12), is realized as in [13] as a *generalized transport operator* $\mathcal{H}_{transport}^{\kappa}(t)$, in the terminology of section 6. All in all, $\mathcal{H}^{\kappa}(t) :=$

$\mathcal{H}_{transport}^\kappa(t) + \mathcal{H}_{nonlocal}^\kappa(t)$ appears to be a time-dependent *bounded perturbation of a generator of contraction operators* in $L^1(\Pi_{b_{max}})$. In our setting, a theorem due originally to T. Kato (see [16], or [27], Theorem 4.41, or [25], §5.3) states that the backward time-inhomogeneous evolution system $\frac{dh(t)}{dt} = \mathcal{H}^\kappa(t)h(t)$ has a unique solution of the form $h(s) = U^\kappa(s, t)h(t)$ ($s \leq t$), where (i) $U^\kappa(s, t)$ is strongly continuous in s and t , $U^\kappa(t, t) = \text{Id}$ and $|||U^\kappa(s, t)|||_{L^1(\Pi_{b_{max}}) \rightarrow L^1(\Pi_{b_{max}})} \leq e^{C(t-s)}$; (ii) $U^\kappa(s, t) = U^\kappa(s, r)U^\kappa(r, t)$ ($s \leq r \leq t$); (iii) $\partial_s U^\kappa(s, t) = \mathcal{H}^\kappa(s)U^\kappa(s, t)$; (iv) $\partial_t U^\kappa(s, t) = -U^\kappa(s, t)\mathcal{H}^\kappa(t)$, the last two equalities holding in the sense of strong topology when evaluated on the domain of definition of the operators. See Remark in §3.3 for a suggested interpretation of the *adjoint* operator $\mathcal{L}_{nonlocal}^\kappa = -(\mathcal{H}_{nonlocal}^\kappa)^\dagger$ as the generator of a generalized, *signed* Markov process, generalizing [7], §4.2, which is due to the fact that $g_{nonlocal}^\kappa$ is a *signed* function. By the usual duality between positivity-preserving generators of random processes and L^1 -contracting Fokker-Planck operators, it is tempting to interpret h_t as the *density* at time t of a random, $\Pi_{b_{max}}$ -valued process. However, h_t is signed, which is incompatible with final estimates based on positivity and inequalities. Hence in practice, one uses the Green function expansion,

$$U(t, T) = U_{transport}(t, T) - \int_t^T ds U(t, s) \mathcal{H}_{nonlocal}(s) U_{transport}(s, T) \quad (1.19)$$

– note that we intentionally left out the precise dependence on the κ -indices, which will be made explicit only in §4.2. With this in mind, one sees that the time-evolution is essentially controlled by $U_{transport}$, which allows one to mimic the strategy of proof of Israelsson leading to (1.10). In the course of the proof, it appears to be necessary to control also the $L^\infty(\Pi_{b_{max}})$ -norms, so we must actually work with semi-groups in $L^1(\Pi_{b_{max}}) \cap L^\infty(\Pi_{b_{max}})$, which does not however deeply modify the above scheme.

In a work in preparation [30], we compute the law of the fluctuation process for V analytic. The derivation does not follow at all Bender's path. Not surprisingly, we are not able to solve explicitly our generalized characteristic equations. However, we remark that $\text{Cov}(Y_{t_1}(x_1), Y_{t_2}(x_2))$ comes out as the *boundary value* of $\text{Cov}(U_{t_1}(z_1), U_{t_2}(z_2))$. In this boundary value limit, most terms (including the non-local term) in $\mathcal{H}(t)$ disappear, and one may show *in the stationary regime* that the two-point function satisfies a simple PDE which can be solved explicitly. Due to the length of the present article, we chose to discuss this in a separate article.

1.2 Notations and basic facts

In this paragraph, we simply assume that V is convex and $V(x) \xrightarrow{|x| \rightarrow +\infty} +\infty$.

Definition 1.1. 1. Let

$$X_t^N := \frac{1}{N} \sum_{i=1}^N \delta_{\lambda_t^i} \quad (1.20)$$

be the empirical measure process.

2. Call

$$Y_t^N := N(X_t^N - X_t) \quad (1.21)$$

the finite N fluctuation process.

The case developed in [13] and [2] is the *harmonic case*, $V(x) = \frac{1}{2}x^2$ (up to normalization), to which we shall often refer. Apart from this very particular case, classical examples include the Landau-Ginzburg potential $V(x) = \frac{1}{2}x^2 + \frac{\lambda}{4}x^4$, $\lambda \geq 0$, for which the support of the equilibrium measure is connected, and the density is the product of the (rescaled) semi-circle law by some explicit polynomial of degree 2 (see e.g. [14], p. 164).

It is proved in (Li-Li-Xie [18], Theorem 1.3) that, provided $X_0^N \xrightarrow{N \rightarrow \infty} \rho_0$, a deterministic density, the empirical measure process (X_t^N) converges in law to a deterministic measure process $(X_t)_{t \geq 0}$ with density ρ_t solution of the non-linear Fokker-Planck equation,

$$\frac{\partial \rho_t(x)}{\partial t} = \frac{\partial}{\partial x} \left(\left(V'(x) - \frac{\beta}{2} p.v. \int \frac{dy}{x-y} \rho_t(y) \right) \rho_t(x) \right) \quad (1.22)$$

with initial condition ρ_0 . Equivalently, for any test function f ,

$$\begin{aligned} \frac{d}{dt} \langle X_t, f \rangle &= \frac{d}{dt} \int \rho_t(x) f(x) dx = - \int V'(x) f'(x) X_t(dx) + \\ &+ \frac{\beta}{4} \int \int \frac{f'(x) - f'(y)}{x-y} X_t(dx) X_t(dy). \end{aligned} \quad (1.23)$$

The equilibrium measure μ_{eq}^N , see (1.3) converges weakly when $N \rightarrow \infty$ to the stationary, deterministic solution $X_t(dx) = \rho_{eq}(x)dx$ of (1.22), where ρ_{eq} is the solution of the following integral equation, called *cut equation*, [14]

$$p.v. \int \frac{\rho_{eq}(x) dx}{x-y} = -\frac{2}{\beta} V'(y) \quad (1.24)$$

Formula (1.23) is formally obtained as in [26] by taking the limit $N \rightarrow \infty$ in the finite N Itô formula (eq. (3) in [13]),

$$\begin{aligned} d \langle X_t^N, f \rangle &= \left(\frac{\beta}{4} \int \int \frac{f'(x) - f'(y)}{x-y} X_t^N(dx) X_t^N(dy) - \int V'(x) f'(x) X_t^N(dx) \right) dt \\ &+ \frac{1}{2} \left(1 - \frac{\beta}{2} \right) \frac{1}{N} \langle X_t^N, f'' \rangle dt + \frac{1}{N\sqrt{N}} \sum_{i=1}^N f'(\lambda_t^i) dW_t^i. \end{aligned} \quad (1.25)$$

Roughly speaking, both terms in the second lign of (1.25) are $O(\frac{1}{N})$ (the argument for the martingale term relies on an L^2 -bound based on the independence of the $(W^i)_{1 \leq i \leq N}$).

Definition 1.2 (Stieltjes transform). *Fix $z \in \mathbb{C} \setminus \mathbb{R}$.*

(i) *Let $f_z(x) := \frac{1}{x-z}$ ($x \in \mathbb{R}$).*

(ii) *Let, for $z \in \mathbb{C} \setminus \mathbb{R}$,*

$$M_t^N := \langle X_t^N, f_z \rangle = \sum_{i=1}^N \frac{1}{\lambda_i(t) - z} \quad (1.26)$$

and

$$M_t := \langle X_t, f_z \rangle = \int \frac{\rho_t(x)}{x-z} dx \quad (1.27)$$

be the Stieltjes transform of X_t^N , resp. X_t .

Starting from the cut equation (1.24) and applying Plemelj's formula (see section 7), one finds *at equilibrium*

$$M(x + i0) = -\frac{2}{\beta}V'(x) + i\pi\rho_{eq}(x) \quad (1.28)$$

$$M(x + i0) - M(x - i0) = 2i\pi\rho_{eq}(x), \quad M(x + i0) + M(x - i0) = -\frac{4}{\beta}V'(x). \quad (1.29)$$

A PDE for the Stieltjes transform of X_t is determined easily from (1.23),

$$\frac{\partial M_t}{\partial t} = \frac{\partial}{\partial z} \left(\frac{\beta}{4}(M_t(z))^2 + V'(z)M_t(z) + T(z) \right), \quad (1.30)$$

where

$$T(z) := \int \frac{V'(x) - V'(z)}{x - z} X_t(dx). \quad (1.31)$$

In the harmonic case, T is simply a constant, hence M is the solution of a complex Burgers equation on $\mathbb{C} \setminus \mathbb{R}$, see e.g. [13], eq. (6). However, this is no more the case in our general setting, and the non-local term T in the right-hand side prevents any explicit solution of the equation. Yet the Stieltjes transform will turn out to be a very convenient technical tool in the computations.

The first idea, coming from [13], is to *transfer the drift in the time-evolution of Y_t^N to the test function f* . This is done through a *straightforward generalization* of (Israelsson [13], Lemma 1):

Proposition 1.3. (see Israelsson [13]) *Assume the following event holds for some constant $R > 0$,*

$$\Omega_R : \sup_{0 \leq t \leq T} \max_{i=1, \dots, N} |\lambda^i| \leq R; \quad \forall t \leq T, \text{ supp}(X_t) \subset [-R, R], \quad (1.32)$$

i.e. that the support of the random point measure X_t^N and of the measure X_t is $\subset [-R, R]$ for $0 \leq t \leq T$. Let $(f_t)_{0 \leq t \leq T}$, $f_t : \mathbb{R} \rightarrow \mathbb{R}$ be such that

$$\frac{\partial f_t}{\partial t}(x) = V'(x)f'_t(x) - \frac{\beta}{4} \int \frac{f'_t(x) - f'_t(y)}{x - y} (X_t^N(dy) + X_t(dy)) \quad (1.33)$$

for all $|x| \leq R$. Then

$$d\langle Y_t^N, f_t \rangle = \frac{1}{2} \left(1 - \frac{\beta}{2}\right) \langle X_t^N, f_t'' \rangle dt + \frac{1}{\sqrt{N}} \sum_{i=1}^N f'_t(\lambda_t^i) dW_t^i. \quad (1.34)$$

As emphasized in the above Proposition, (1.33) need only hold on $[-R, R]$, because $\langle X_t^N, f \rangle$ and $\langle X_t, f \rangle$ do *not* depend on the values of f on $\mathbb{C} \setminus [-R, R]$. This remark is useless in the harmonic case, but will help us in this work separate the rather subtle effect due to the $(\frac{1}{x})$ -kernel on $[-R, R]$ from the not so trivial effect for large x of the confining potential V .

The above Proposition is a direct consequence of Itô's formula applied to the fluctuation process (just subtract (1.25) from (1.23)),

$$\begin{aligned} d\langle Y_t^N, f \rangle &= \frac{\beta}{4} \int \int \frac{f'(x) - f'(y)}{x - y} [X_t^N(dx) + X_t(dx)] Y_t^N(dy) - \int V'(x) f'(x) Y_t^N(dx) \\ &\quad + \frac{1}{2} \left(1 - \frac{\beta}{2}\right) \langle X_t^N, f'' \rangle + \frac{1}{\sqrt{N}} \sum_{i=1}^N f'(\lambda_t^i) dW_t^i. \end{aligned} \quad (1.35)$$

Then Israelsson solves eq. (1.33) in the harmonic case by using as test functions the $\frac{c}{\cdot - z}$, $c \in \mathbb{C}, z \in \mathbb{C} \setminus \mathbb{R}$, on which the generator of time-evolution acts in a particularly simple way. We do not reproduce their results here however, since they do not separate the analysis of the term due to the harmonic potential from that due to the two-body logarithmic potential. We shall analyze (1.33) in section 3 after we have introduced Stieltjes decompositions.

Normalization: the reader willing to compare our results with those of Israelsson [13] or Bender [2] should take into account the different choices of normalization. Compared to [13], we fix $\sigma = 1$ and let $\alpha = \frac{\beta}{2}$, $\gamma = \frac{1}{2}(1 - \frac{\beta}{2})$. After rescaling the λ^i 's by a factor $\beta^{-1/2}$, we obtain for V quadratic [2] with $\sigma = \frac{1}{2}$.

1.3 Main result and outline of the article

Assumptions on V .

We assume V to satisfy the following conditions:

- (i) V is C^{11} ;
- (ii) $V'(0) = 0$ and the functions $x \mapsto V(\pm\sqrt{|x|})$ are convex;
- (iii) $V(x), |V'(x)| \rightarrow_{|x| \rightarrow \infty} +\infty$;
- (iv) $\limsup_{x \rightarrow +\infty} \left| \frac{V'(x)}{V'(-x)} + 1 \right| < 1$;
- (v) V has regular variation in the following sense: there exist $x_V^0 > 0$ and $0 < c \leq \frac{1}{2}$ such that $|\frac{V(y)}{V(x)} - 1| \leq \frac{1}{2}$ and $|\frac{V'(y)}{V'(x)} - 1| \leq \frac{1}{2}$ for all $x, y \in \mathbb{R}$ such that $|x|, |y| \geq x_V^0$ and $|\frac{y}{x} - 1| \leq c$, $i = 0, 1, 2, 3$.
- (vi) V has polynomial bound in the following sense:

$$\begin{aligned} \limsup_{R \rightarrow +\infty} \frac{\sum_{j=0}^{11} R^j \sup_{[-R, R]} |V^{(j)}(x)|}{V(R)} &< \infty, \\ \limsup_{R \rightarrow +\infty} \frac{\sum_{j=0}^{10} R^j \sup_{[-R, R]} |V^{(1+j)}(x)|}{|V'(R)|} &< \infty. \end{aligned} \quad (1.36)$$

Main examples are convex polynomials, or suitable, smooth perturbations thereof; for a convex function V such that $V(x) \rightarrow_{|x| \rightarrow \infty} +\infty$, we can arrange that $V'(0) = 0$ (see (ii))

by translating $x \rightarrow x - x_{\min}$, where x_{\min} is the location of a local minimum of V (say, of the absolute minimum if it is unique). The *square-root* in (ii) comes from the Lyapunov inequality found for the average of the *squares* of the eigenvalues in §5.2. Condition (iii) is a *confinement condition*; the hypothesis $|V'(x)| \rightarrow_{|x| \rightarrow \infty} +\infty$ is used in a weak form in the technical arguments below (5.51). The remaining hypotheses (iv), (v), (vi) are somewhat ad-hoc, and other sets of hypotheses are perfectly possible, depending on what kinds of potentials one wants to cover. Condition (iv) – stating that V is roughly symmetric – is used in §5.2, together with the square-root condition (ii). Conditions (v) and (vi) come from the necessity of controlling the exponential time growth of the linear operator appearing in Proposition 1.3, whose rate increases in principle with the support of the random point measure, by the probability that the support is large. In our approach, we deal with this problem by using smooth *scaled* cut-offs, i.e. if the support of the measure is $\subset [-R, R]$, then we use a cut-off $\chi_R(\cdot) = \chi_1(\frac{\cdot}{R})$ such that $\chi_1 \equiv 1$ on Ω_1 and $\chi_1 \equiv 0$ on $\mathbb{R} \setminus \Omega_2$ for some open neighborhoods $\Omega_2 \supsetneq \Omega_1 \supsetneq [-1, 1]$ (see Definition 3.3). This explains the general form of conditions (v), (vi), but our precise condition (vi) results from taking into account all the terms in our expansion and comparing them with the large deviation statement proved in section 5 (see Theorem 5.1). By looking carefully at our computations, the reader will realize that the scaled decay assumed in (1.36), although natural for polynomial-like functions, is much too demanding, even for this kind of cut-off. It may well be that our results extend to convex potentials V increasing faster than polynomials (e.g. exponentials), but it would clearly require choosing other types of cut-offs, e.g. cut-offs *without* scaling, and presumably also, use a cut-off scale b_{\max} depending on V and possibly on R (see section 2). It is also clear that asymptotic bounds for the distribution of the eigenvalues, e.g. bounds for the density, would help greatly enhance the class of admissible potentials, but this would require much more work.

Under assumptions (i), (ii), it is a simple exercise to show that V itself is convex. Hence (see e.g. [14], Theorem 2.1 and Proposition 3.1) the equilibrium measure ρ_{eq} is well-defined and compactly supported, its support $[a, b]$ is connected, and ρ_{eq} is a solution of the cut-equation (1.24). We denote by $M_{eq}(z) := \int dx \frac{\rho_{eq}(x)}{x-z}$ ($z \in \mathbb{C} \setminus \mathbb{R}$) its Stieltjes transform.

Assumptions on the initial measure.

Let $\mu_0^N = \mu_0(\{\lambda_0^i\}_i)$ be the initial measure of the stochastic process $\{\lambda_t^i\}_{t \geq 0, i=1, \dots, N}$, and $X_0^N := \frac{1}{N} \sum_{i=1}^N \delta_{\lambda_0^i}$ be the initial empirical measure. Since N varies, we find it useful here to add an extra upper index $(\lambda_0^{N,i})_{i=1, \dots, N}$ to denote the initial condition of the process for a given value of N . We assume that:

- (i) (large deviation estimate for the initial support) there exist some constant $c, R_V > 0$ and $N_0 \in \mathbb{N}^*$ such that, for $N \geq N_0$ and $R \geq R_V$,

$$\mathbb{P}[\max_{i=1, \dots, N} |\lambda_0^{N,i}| > R] \leq e^{-cNV(R)}. \quad (1.37)$$

- (ii) $X_0^N \xrightarrow{law} \rho_0(x) dx$ when $N \rightarrow \infty$, where $\rho_0(x)$ is a deterministic measure;

(iii) (rate of convergence)

$$(\mathbb{E}[|M_0^N(z) - M_0(z)|^4])^{1/4} = O(\frac{1}{Nb}) \quad (1.38)$$

for $z = a + ib \in \mathbb{C} \setminus \mathbb{R}$, where $M_0(z) := \int dx \frac{\rho_0(x)}{x-z}$ is the Stieltjes transform of ρ_0 .

Section 5 yields a somewhat ad-hoc but very reasonable entropic-type condition on the N -dependent initial conditions $(\lambda_0^{N,i})_{i=1,\dots,N}$ ensuring that the large deviation estimate (i) holds (see Lemma 5.4). Whether the latter condition holds or not, in any case, we prove in Theorem 5.1 that the initial large deviation estimate (i) implies a uniform-in-time large deviation estimate for the support of the random point measure, which is essential for our main result.

Definition 1.4 (Sobolev spaces). Let $H_n := \{f \in L^2(\mathbb{R}) \mid \|f\|_{H_n} < \infty\}$ ($n \geq 0$), where $\|f\|_{H_n} := (\int d\xi (1 + |\xi|^2)^{n/2} |\mathcal{F}f(\xi)|^2)^{1/2}$, and $H_{-n} := (H_n)'$ its dual.

The measure-valued process Y^N may be shown to converge in $C([0, T], H_{-6})$:

Main Theorem (Gaussianity of limit fluctuation process).

Let Y_t^N be the finite n fluctuation process (see Definition 1.1). Then:

1. $Y^N \xrightarrow{\text{law}} Y$ when $N \rightarrow \infty$, where Y is a Gaussian process. More precisely, Y^N converges to Y weakly in $C([0, T], H_{-8})$;
2. let $\phi_h(Y_t^N) := e^{i\langle Y_t^N, \mathcal{C}^0 h \rangle}$, with \mathcal{C}^0 (Stieltjes decomposition of order 0 with arbitrary cut-off $b_{\max} > 0$) as in Definition 2.3. Then (see (4.7))

$$\mathbb{E}[\phi_{h_T}(Y_T) | \mathcal{F}_t] = \phi_{h_t}(Y_t) \mathbb{E} \left[\exp \left(\frac{1}{2} \int_t^T \left[i \left(1 - \frac{\beta}{2} \right) \langle X_s, (\mathcal{C}^0 h_s)'' \rangle - \langle X_s, ((\mathcal{C}^0 h_s)')^2 \rangle \right] ds \right) \right] \quad (1.39)$$

The article is organized as follows. We first introduce a family of *Stieltjes decompositions* \mathcal{C}^κ depending on a regularity index $\kappa = 0, 1, 2, \dots$ (see section 2). The main section is section 3, where we rewrite the r.h.s. of eq. (1.33) using Stieltjes decompositions as a sum of linear operators which we call *generators*; these are of two types: generalized transport operators summing up to $\mathcal{H}_{\text{transport}}$, and bounded operators summing up to $\mathcal{H}_{\text{nonlocal}}$. We prove our Main Theorem in section 4. Since (1.39) is formally just a consequence of Itô's formula, and most of the technical arguments used in Israelsson's paper to justify this formula hardly depend on V , section 4 really revolves around a fundamental estimate, Lemma 4.2, which is based on properties of the characteristics, hence is strongly V -dependent. The analysis of the generators made in details in section 3 allows one to prove the latter estimate, with a constant prefactor which increases exponentially with the support of the measure. To conclude, one uses as input *large deviation estimates for the support of the measure* proved in section 5. We did not comment very much in this Introduction about this large deviation estimates, because it lies somewhat out of the mainstream of the article; it may be read independently; however, it is an interesting result in itself. Finally, sections 6 and 7 are appendices, where we collected some well-known facts and formulas about transport equations and Stieltjes transforms.

2 Stieltjes decompositions

In (Israelsson [13], Lemma 9) one finds the following decomposition of an L^1 function $f : \mathbb{R} \rightarrow \mathbb{R}$ living in the Sobolev space H_2 as a sum of functions of the type

$$f_z : x \mapsto \frac{1}{x - z}, \quad (2.1)$$

where $z = a + ib$, $b \neq 0$ (see section 7),

$$f(x) = \int_{-\infty}^{+\infty} da \int_{-\infty}^{+\infty} (-ib) db f_z(x) h(a), \quad h(a) = -f''(a). \quad (2.2)$$

The above "reproducing kernel type" decomposition is clearly not unique. The proof is based on the fact that, for $\kappa = 0, 1, 2, \dots$ (see (7.11))

$$\begin{aligned} \int_{-\infty}^{+\infty} (-ib) db |b|^\kappa \cdot \mathcal{F}(f_{ib})(s) &= 2 \int_0^{+\infty} db |b|^\kappa \mathcal{F}(\text{Im } f_{ib})(s) \\ &= 2\pi \int_{-\infty}^{+\infty} db |b|^{1+\kappa} e^{-b|s|} = 2\pi \cdot (1 + \kappa)! |s|^{-2-\kappa}, \end{aligned} \quad (2.3)$$

from which we also get the following family of decompositions, valid for $f \in L^1 \cap H_{2+\kappa}$, $\kappa \in \mathbb{N}$,

$$f(x) = \int_{-\infty}^{+\infty} da \int_{-\infty}^{+\infty} (-ib) db \frac{|b|^\kappa}{(1 + \kappa)!} f_z(x) h(a), \quad h(a) = \mathcal{F}^{-1}(|s|^{2+\kappa} \mathcal{F}(f))(a), \quad (2.4)$$

a straightforward generalization of (2.2) obtained by choosing some arbitrary value of κ instead of $\kappa = 0$ in (2.3). Note that h is *real* since $\mathcal{F}^{-1}(|s| \cdot)$ is given by a *real-valued* convolution kernel (see section 7).

The reason for introducing this κ -dependent family of decompositions is that the coefficient of $f_z(x)$ now *vanishes to order $1 + \kappa$ instead of 1 on the real axis*, a property inherited from the assumed supplementary regularity of f . Note that, for κ *even*, $\mathcal{F}^{-1}(|s|^{2+\kappa} \mathcal{F}(\cdot))$ is the differential operator $(-\partial_s^2)^{1+\kappa/2}$. For κ *odd*, on the other hand, one gets derivatives of the $(\frac{1}{x})$ -kernel (see section 7).

Since all interesting phenomena appear when $|b|$ is small, and we want to avoid artificial problems arising when $|b|$ is not bounded, we shall actually use analogous decompositions in which $|b|$ ranges from 0 to some maximal value $b_{max} > 0$. This introduces the following changes. First, instead of (2.2), we get

$$f(x) = \int_{-\infty}^{+\infty} da \int_{-b_{max}}^{b_{max}} (-ib) db \frac{|b|^\kappa}{(1 + \kappa)!} f_z(x) h(a), \quad h(a) = (\mathcal{F}^{-1}(K_{b_{max}}^\kappa) * f)(a), \quad (2.5)$$

where

$$K_{b_{max}}^\kappa(s) := \left(2 \int_0^{b_{max}} db |b|^{1+\kappa} \cdot \mathcal{F}(\text{Im } (f_{ib}))(s) \right)^{-1} \quad (2.6)$$

(note that the above integral is > 0 by (7.11)). We now study the convolution operator

$$\mathcal{K}_{b_{max}}^\kappa : f \mapsto \mathcal{F}^{-1}(K_{b_{max}}^\kappa) * f, \quad (2.7)$$

depending on the parity of κ :

(i) For κ even,

$$\int_0^{b_{max}} db |b|^{1+\kappa} \cdot \mathcal{F}(\text{Im } f_{ib})(s) = \pi \int_0^{b_{max}} db |b|^{1+\kappa} e^{-b|s|} = \pi (-\partial_s^2)^{\kappa/2} (k_{b_{max}}^0(|s|)), \quad (2.8)$$

where

$$k_{b_{max}}^0(|s|) = \frac{1}{s^2} \left(1 - (1 + b_{max}|s|) e^{-b_{max}|s|} \right). \quad (2.9)$$

When $|s| \rightarrow \infty$, $k_{b_{max}}^0(|s|) \sim s^{-2}$; on the other hand, $k_{b_{max}}^0(|s|) \stackrel{s \rightarrow 0}{\sim} b_{max}^2 \sum_{k \geq 0} \frac{(-1)^k}{(k+2)!} (k+1)(b_{max}|s|)^k$. Thus $(-\partial_s^2)^{\kappa/2} (k_{b_{max}}^0(|s|)) \stackrel{|s| \rightarrow \infty}{\sim} (-1)^{\kappa/2} \frac{s^{-(2+\kappa)}}{(\kappa+1)!}$, and $(-\partial_s^2)^{\kappa/2} (k_{b_{max}}^0(|s|)) \stackrel{|s| \rightarrow 0}{\sim} (-1)^{\kappa/2} \frac{b_{max}^{2+\kappa}}{2+\kappa}$. It is a simple exercise to prove the following: let

$$\underline{K}_{b_{max}}^\kappa(s) := (-1)^{\kappa/2} \left((\kappa+1)! s^{2+\kappa} + (2+\kappa) b_{max}^{-(2+\kappa)} \right)^{-1} K_{b_{max}}^\kappa(s) - 1. \quad (2.10)$$

Then $(\underline{K}_{b_{max}}^\kappa)^{(j)}(s)$, $j = 0, 1, 2$ is $O(\frac{b_{max}^j}{1+b_{max}^2 s^2})$ uniformly in s and b_{max} . Hence the convolution operator

$$\underline{K}_{b_{max}}^\kappa : f \mapsto \mathcal{F}^{-1}(\underline{K}_{b_{max}}^\kappa) * f \quad (2.11)$$

is a bounded operator from $L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ to $L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$. Indeed,

$$|||\underline{K}_{b_{max}}^\kappa|||_{L^1(\mathbb{R}) \rightarrow L^1(\mathbb{R})}, |||\underline{K}_{b_{max}}^\kappa|||_{L^\infty(\mathbb{R}) \rightarrow L^\infty(\mathbb{R})} \leq ||\mathcal{F}^{-1}(\underline{K}_{b_{max}}^\kappa)||_{L^1(\mathbb{R})}$$

and

$$\begin{aligned} |\mathcal{F}^{-1}(\underline{K}_{b_{max}}^\kappa)(x)| &\leq \min \left(||\underline{K}_{b_{max}}^\kappa||_{L^1}, \frac{1}{x^2} ||(\underline{K}_{b_{max}}^\kappa)''||_{L^1} \right) \\ &= O \left(\int \frac{ds}{1+b_{max}^2 s^2} \right) \cdot \min(1, (\frac{b_{max}}{x})^2) \\ &= \frac{1}{b_{max}} \cdot \min(1, (\frac{b_{max}}{x})^2), \end{aligned} \quad (2.12)$$

from which $|||\underline{K}_{b_{max}}^\kappa|||_{L^1(\mathbb{R}) \rightarrow L^1(\mathbb{R})}, |||\underline{K}_{b_{max}}^\kappa|||_{L^\infty(\mathbb{R}) \rightarrow L^\infty(\mathbb{R})} = O(1)$. We may therefore write

$$\mathcal{K}_{b_{max}}^\kappa = (1 + \underline{K}_{b_{max}}^\kappa) \left(-(\kappa+1)! \partial_x^{2+\kappa} + (-1)^{\kappa/2} (2+\kappa) b_{max}^{-(2+\kappa)} \right). \quad (2.13)$$

(ii) For κ odd,

$$\int_0^{b_{max}} db |b|^{1+\kappa} \cdot \mathcal{F}(\text{Im } f_{ib})(s) = \pi \int_0^{+\infty} db |b|^{1+\kappa} e^{-b|s|} = \pi (-\partial_s^2)^{(\kappa+1)/2} (k_{b_{max}}^1(|s|)), \quad (2.14)$$

where

$$k_{b_{max}}^1(|s|) = \frac{1}{|s|} (1 - e^{-b_{max}|s|}). \quad (2.15)$$

When $|s| \rightarrow \infty$, $k_{b_{max}}^1(|s|) \sim |s|^{-1}$; on the other hand, $k_{b_{max}}^1(|s|) \stackrel{s \rightarrow 0}{\sim} b_{max} \sum_{k \geq 0} \frac{(-1)^k}{(k+1)!} (b_{max}|s|)^k$. Thus $(-\partial_s^2)^{(\kappa+1)/2} (k_{b_{max}}^1(|s|)) \stackrel{|s| \rightarrow \infty}{\sim} (-1)^{(\kappa+1)/2} \frac{|s|^{-1} s^{-(1+\kappa)}}{(\kappa+1)!}$, and $(-\partial_s^2)^{(\kappa+1)/2} (k_{b_{max}}^1(|s|)) \stackrel{|s| \rightarrow 0}{\sim}$

$(-1)^{(\kappa+1)/2} \frac{b_{max}^{2+\kappa}}{2+\kappa}$. Therefore the previous analysis, from (2.10) to the line before (2.13), remains valid, with $(-1)^{\kappa/2}$, resp. $s^{2+\kappa}$, replaced with $(-1)^{(\kappa+1)/2}$, resp. $|s|s^{1+\kappa}$, and one obtains using (7.20):

$$\mathcal{K}_{b_{max}}^\kappa = (1 + \underline{\mathcal{K}}_{b_{max}}^\kappa) \left(-(\kappa+1)! \, iH\partial_x^{2+\kappa} + (-1)^{(\kappa+1)/2} (2+\kappa) b_{max}^{-(2+\kappa)} \right) \quad (2.16)$$

where H is the Hilbert transform, defined by the principal value integral

$$Hf(x) := \frac{1}{\pi} p.v. \int_{-\infty}^{+\infty} \frac{1}{x-y} f(y) dy \quad (2.17)$$

and $\underline{\mathcal{K}}_{b_{max}}^\kappa$, defined by analogy with (i) as the convolution with respect to the inverse Fourier transform of

$$\underline{K}_{b_{max}}^\kappa(s) = (-1)^{(\kappa+1)/2} \left((\kappa+1)! |s|^{2+\kappa} + (2+\kappa) b_{max}^{-(2+\kappa)} \right)^{-1} K_{b_{max}}^\kappa(s) - 1, \quad (2.18)$$

has operator norm $O(1)$ on $L^1(\mathbb{R})$ and on $L^\infty(\mathbb{R})$.

The above formulas are particular instances of *Stieltjes decompositions*, where $h = h(a, b)$ is allowed to be complex-valued and to depend on b .

Definition 2.1 (upper half-plane). 1. Let $\Pi^+ := \{z \in \mathbb{C} \mid \text{Im}(z) > 0\}$.

2. For $b_{max} > 0$, let $\Pi_{b_{max}}^+ := \{z \in \mathbb{C} \mid 0 < \text{Im}(z) < b_{max}\}$.

3. Let $\Pi^- := -\Pi^+$, $\Pi_{b_{max}}^- := -\Pi_{b_{max}}^+$ and $\Pi := \Pi^+ \uplus \Pi^-$, $\Pi_{b_{max}} := \Pi_{b_{max}}^+ \uplus \Pi_{b_{max}}^-$.

Definition 2.2. Let, for $p \in [1, +\infty]$ and $b_{max} > 0$,

$$L^p(\Pi_{b_{max}}) := \{h : \Pi_{b_{max}} \rightarrow \mathbb{R} \mid h(\bar{z}) = \overline{h(z)} \ (z \in \Pi_{b_{max}}^+) \text{ and } \|h\|_{L^p(\Pi_{b_{max}})} < \infty\}, \quad (2.19)$$

where

$$\|h\|_{L^p(\Pi_{b_{max}})} := \left(\int_{-\infty}^{+\infty} da \int_{-b_{max}}^{b_{max}} db |h(a, b)|^p \right)^{1/p} \quad (p < \infty), \quad \|h\|_{L^\infty(\Pi_{b_{max}})} := \sup_{z \in \Pi_{b_{max}}} |h(z)|. \quad (2.20)$$

We will be mostly interested in the extreme cases $p = 1$, $p = \infty$. Letting formally $b_{max} \rightarrow +\infty$ one obtains in an obvious way the space $L^p(\Pi)$ with its norm $\|\cdot\|_{L^p(\Pi)}$. However, we shall actually fix some finite value of b_{max} , say (for convenience only), $0 \leq b_{max} \leq \frac{1}{2}$, implying: $\ln(1/|b|) \geq \ln 2 > 0$.

Definition 2.3 (Stieltjes decomposition). Let $\kappa = 0, 1, 2, \dots$

1. Let $h \in L^1(\Pi_{b_{max}})$. We say that $f : \mathbb{R} \rightarrow \mathbb{R}$ has Stieltjes decomposition h of order κ and cut-off b_{max} on $[-R, R]$ if, for all $|x| \leq R$,

$$f(x) = (\mathcal{C}^\kappa h)(x) := \int_{-\infty}^{+\infty} da \int_{-b_{max}}^{b_{max}} (-ib) db \frac{|b|^\kappa}{(1+\kappa)!} f_z(x) h(a, b). \quad (2.21)$$

2. The function $h : (a, b) \mapsto \mathcal{K}_{b_{max}}^\kappa(f)(a)$ is called the standard Stieltjes decomposition of order κ and cut-off b_{max} of f .

Thanks to the symmetry condition, $h(\bar{z}) = \overline{h(z)}$, (2.21) may be rewritten in the form

$$(\mathcal{C}^\kappa h)(x) = 2 \int_{-\infty}^{+\infty} da \int_0^{b_{max}} db \frac{|b|^{1+\kappa}}{(1+\kappa)!} \operatorname{Im} [f_z(x)h(a, b)], \quad (2.22)$$

from which it is apparent that f is indeed real-valued.

As already emphasized before, Stieltjes decompositions are not unique. In fact, it turns out to be useful to introduce a larger family of decompositions depending on a further scale parameter, $\rho > 0$. We shall give less details since we apply these to the functions $\operatorname{Im} f_{z_T}$ with $\operatorname{Im} z_T > 0$ only, and restrict to $\kappa = 0$. For $0 < \rho \leq b_{max}$, we write

$$K_{b_{max}, \rho}^0(s) := \left(\int_0^{b_{max}} db b e^{-b/\rho} \cdot \mathcal{F}(\operatorname{Im} (f_{ib}))(s) \right)^{-1} \quad (2.23)$$

and let as before

$$\mathcal{K}_{b_{max}, \rho}^0 : f \mapsto \mathcal{F}^{-1}(K_{b_{max}, \rho}^0) * f, \quad (2.24)$$

so that, for $b_T \neq 0$,

$$\operatorname{Im} [f_{ib_T}](x) = \int_{-\infty}^{+\infty} da \int_0^{b_{max}} db b e^{-b/\rho} \operatorname{Im} [f_z(x)] \mathcal{K}_{b_{max}, \rho}^0(\operatorname{Im} f_{ib_T})(a). \quad (2.25)$$

Then (compare with (2.8)) $(K_{b_{max}, \rho}^0)^{-1} = \pi k_{b_{max}}^0(\frac{1}{\rho} + |s|)$. Thus (emphasizing only the differences with $K_{b_{max}}^0 = \lim_{\rho \rightarrow +\infty} K_{b_{max}, \rho}^0$) $k_{b_{max}, \rho}^0(|s|) \stackrel{s \rightarrow 0}{\sim} C\rho^2$ instead of b_{max}^2 , with $C := 1 - (1 + b_{max}/\rho)e^{-b_{max}/\rho}$ bounded away from 0. Hence

$$\begin{aligned} \underline{K}_{b_{max}, \rho}^0(s) &:= (|s| + \rho^{-1})^{-2} K_{b_{max}, \rho}^0 - 1 \\ &= - \frac{(1 + b_{max}(|s| + \rho^{-1}))e^{-b_{max}(|s| + \rho^{-1})}}{1 - (1 + b_{max}(|s| + \rho^{-1}))e^{-b_{max}(|s| + \rho^{-1})}} \end{aligned} \quad (2.26)$$

is a $O(e^{-\frac{1}{2}b_{max}(|s| + \rho^{-1})})$, defining a bounded operator on $L^1 \cap L^\infty$ with operator norm

$$O\left(\int ds e^{-\frac{1}{2}b_{max}(|s| + \rho^{-1})}\right) \cdot \int dx \min(1, \frac{b_{max}}{x})^2 = O(e^{-\frac{1}{2}b_{max}/\rho}) \quad (2.27)$$

(compare with (2.12)). On the other hand, letting $f = \operatorname{Im} [f_{z_T}]$, with $z_T = a_T + ib_T$, $b_T > 0$, we find using (7.4), (7.11)

$$\mathcal{F}^{-1}((|s| + \rho^{-1})^2 \mathcal{F} \operatorname{Im} [f_{z_T}])(a) = \left(\rho^{-1} - \frac{\partial}{\partial b_T} \right)^2 \left(\frac{b_T}{(a - a_T)^2 + |b_T|^2} \right) \quad (2.28)$$

Hence, letting

$$h(a, b) := e^{-b/\rho} \mathcal{K}_{b_{max}, \rho}^0(\operatorname{Im} f_{z_T})(a) = e^{-b/\rho} (1 + \underline{K}_{b_{max}, \rho}^0) \mathcal{F}^{-1}((|s| + \rho^{-1})^2 \mathcal{F} f_{z_T})(a), \quad (2.29)$$

we get:

$$\begin{aligned} \int_{-\infty}^{+\infty} da \int_{-b_{max}}^{b_{max}} db h(a, b) &= O(1)\rho \left\{ \frac{1}{\rho^2} + \frac{1}{\rho b_T} + \frac{1}{|b_T|^2} \right\} \\ &= O\left(\frac{1}{\rho}(1 + O((\frac{\rho}{b_T})^2))\right) \end{aligned} \quad (2.30)$$

which is minimal, of order

$$\|h\|_{L^1(\Pi_{b_{max}})} \approx b_T^{-1}, \quad (2.31)$$

when $\rho \approx b_T$. Choosing $\rho = b_T/C$ for some large enough absolute constant $C > 0$, we further obtain a *positive* function h , which can hence be interpreted as a *density*. Also, one easily checks that, still with $\rho \approx b_T$,

$$\|(a, b) \mapsto \ln(1/|b|)h(a, b)\|_{L^1(\Pi_{b_{max}})} \approx \ln(1/b_T)b_T^{-1} \quad (2.32)$$

if $0 < b_T \leq \frac{1}{2}$. On the other hand, $\|h\|_{L^\infty(\Pi_{b_{max}})} \approx b_T^{-3}$. All of this plays an important rôle in section 4.

3 Generators

The general purpose of the section is the following: for $\kappa = 0, 1, 2, \dots$ *fixed*, we want to write down an explicit time-dependent operator $\mathcal{H}(t)$ such that the right-hand side of (1.33) for f_t decomposed as

$$f_t(x) = \int_{-\infty}^{+\infty} da \int_{-b_{max}}^{b_{max}} (-ib) db \frac{|b|^\kappa}{(1+\kappa)!} f_z(x) h_t(a, b) \quad (3.1)$$

(see Definition 2.3) is equal to

$$\int_{-\infty}^{+\infty} da \int_{-b_{max}}^{b_{max}} (-ib) db \frac{|b|^\kappa}{(1+\kappa)!} f_z(x) \mathcal{H}(h_t)(t; a, b) \quad (3.2)$$

where $\mathcal{H}(h_t)(t; a, b) \equiv (\mathcal{H}(t)(h_t))(a, b)$.

Given the characteristic evolution in the z -coordinate found by Israelsson (which we recall in §3.1 below) – which one may view as a *deterministic* Markov process – it is natural to think of the function h as of a density $h(a, b)da db$, and then to interpret $\mathcal{H}(t)$ as a *Fokker-Planck operator*, whence (by duality) $\mathcal{L}(t) := (\mathcal{H}(t))^\dagger$ as the generator of a random process (see section 6 for more). However, contrary to the harmonic case studied by Israelsson and Bender, in general we obtain a truly *random* process, furthermore, a *signed* process, with h a signed function. For lack of references on these notions, we shall refrain from developing this signed Markov process interpretation, and solve instead the evolution equation

$$\frac{dh_t}{dt}(a, b) = \mathcal{H}(h_t)(t; a, b) \quad (3.3)$$

using *semi-group theory*.

We have been voluntarily been vague up to this point about the double dependency of \mathcal{H} on the integer index κ and the cut-off scale b_{max} . Why couldn't one just set $\kappa = 0$ and let $b_{max} \rightarrow +\infty$, as does Israelsson ? The reason is, we cannot handle properly the potential-dependent part of the generator (save when V is order ≤ 2 , which is the case considered in [13]) without introducing various cut-offs and perturbative arguments – unless maybe if V is analytic (or even better, polynomial), where another strategy is perhaps possible. Since we do not want to make this assumption, we shall:

- (1) in practice replace $V(x)$ by its *Taylor expansion to order 2 around all points in the support of the measures* $(X_t^N)_{0 \leq t \leq T}$ and $(X_t)_{0 \leq t \leq T}$ and *treat the Taylor remainder as a perturbation*. Since V is not bounded at infinity, it is important *not* to Taylor expand around any point on the real line, but only on a *compact interval*; choosing as compact interval the support is natural because the singular kernel term in (1.33) vanishes outside. *For brevity, we shall henceforth call support of the measure the union* $(\cup_{0 \leq t \leq T} \text{supp}(X_t^N)) \cup (\cup_{0 \leq t \leq T} \text{supp}(X_t))$ *and denote by* $R > 0$ *any number such that the support of the measure is* $\subset [-R, R]$. We rely on the bounds developed in section 5 to argue that the probability of the support not to be included in $[-R, R]$ for some large enough R decreases exponentially fast when $R \rightarrow \infty$. Then we naturally decompose $h \in L^1(\Pi_{b_{max}})$ as $h^{int} + h^{ext}$ where $\text{supp}(h^{in}) \subset [-3R, 3R] \times [-b_{max}, b_{max}]$ and $\text{supp}(h^{ext}) \subset (\mathbb{R} \setminus [-2R, 2R]) \times [-b_{max}, b_{max}]$, for instance by writing $h(a, b) = \bar{\chi}_R(a)h(a, b) + (1 - \bar{\chi}_R(a))h(a, b)$, where $\bar{\chi}_R : \mathbb{R} \rightarrow [0, 1]$ is some smooth function such that $\text{supp}(\bar{\chi}_R) \subset [-3R, 3R]$ and $\text{supp}(1 - \bar{\chi}_R) \subset \mathbb{R} \setminus [-2R, 2R]$. The action of \mathcal{H} on h^{ext} is very simple and can be added to the the action of the remainder term $\mathcal{H}_{nonlocal}$ discussed in (2), so we shall not mention it any more until §3.3.
- (2) in order to be able to treat the part (thereafter denoted by $\mathcal{H}_{nonlocal}$) of the generator due to the remainder term as a perturbation, it is important to see that $\mathcal{H}_{nonlocal}h(a, b) = O(|b|)$ when $b \rightarrow 0$. This being the case, we may also consider $\mathcal{H}_{nonlocal}$ as an operator *intertwining a Stieltjes decomposition of order κ with a Stieltjes decomposition of order $\kappa + 1$* , leading to a modification of the scheme developed around (3.1): namely, we want the right-hand side of (1.33) for f_t decomposed as (3.1) for some integer index κ to be equal to

$$\int_{-\infty}^{+\infty} da \int_{-b_{max}}^{b_{max}} (-ib) db \frac{|b|^{1+\kappa}}{(2+\kappa)!} f_z(x) \mathcal{H}^{\kappa+1, \kappa}(h_t)(t; a, b). \quad (3.4)$$

Thus, instead of a single operator \mathcal{H}^0 , we must deal simultaneously with a family of operators \mathcal{H}^κ and a family of intertwining operators $\mathcal{H}_{nonlocal}^{\kappa+1, \kappa}$, for $\kappa = 0, 1, 2, \dots$. The way to do this properly, using an expansion of the Green kernel, is explained in section 4, as a preparation for the proof of Lemma 4.2 (see in particular Definition 4.3). For the moment we need to understand only the following: starting from a Stieltjes decomposition (3.1) of order $\kappa = 0$, we show that the r.-h.s. of (1.33) is decomposed as

$$\begin{aligned} & \int_{-\infty}^{+\infty} da \int_{-b_{max}}^{b_{max}} (-ib) db f_z(x) \mathcal{H}_{transport}^0(h_t)(t; a, b) \\ & + \int_{-\infty}^{+\infty} da \int_{-b_{max}}^{b_{max}} (-ib) db \frac{|b|}{2} f_z(x) \mathcal{H}_{nonlocal}^{1,0}(h_t)(t; a, b). \end{aligned} \quad (3.5)$$

The operator $\mathcal{H}_{transport}^0(t)$ – corresponding to \mathcal{H} severed from its remainder term –, which turns out to be a *transport operator*, is the generator of a semi-group, i.e. can be integrated. In turn one must consider the action of \mathcal{H} on the second term in (3.5), which is done in a similar way, but with κ -indices incremented by 1. In principle this could be done over and over again, but in practice one may directly integrate the generator $\mathcal{H}^2(t) = \mathcal{H}_{transport}^2(t) + \mathcal{H}_{nonlocal}^2$, so that the maximal value of κ is 2. Schematically, we perform the following operations,

$$\begin{aligned} h_T &\longrightarrow U_{transport}^0(s, T)h_T \longrightarrow \mathcal{H}_{nonlocal}^{1,0}(s)U_{transport}^0(s, T)h_T \\ &\longrightarrow U_{transport}^1(s', s)\mathcal{H}_{nonlocal}^{1,0}(s)U_{transport}^0(s, T)h_T \\ &\longrightarrow \mathcal{H}_{nonlocal}^{2,0}(s')U_{transport}^1(s', s)\mathcal{H}_{nonlocal}^{1,0}(s)U_{transport}^0(s, T)h_T \\ &\longrightarrow U^2(0, s')\mathcal{H}_{nonlocal}^{2,1}(s')U_{transport}^1(s', s)\mathcal{H}_{nonlocal}^{1,0}(s)U_{transport}^0(s, T)h_T \end{aligned} \quad (3.6)$$

and integrate over $0 \leq s' \leq s \leq T$ (see (4.19)).

- (3) In (3.6) above, one must see what happens when h is decomposed as $h^{int} + h^{ext}$. This is easily done with the remainder terms generated by $\mathcal{H}_{nonlocal}$ because they are bounded; one need just apply the $\bar{\chi}_R + (1 - \bar{\chi}_R)$ decomposition to $\mathcal{H}_{nonlocal}h$, where by abuse of notation we write $\bar{\chi}_R$ for $\bar{\chi}_R \otimes 1 : (a, b) \mapsto \bar{\chi}_R(a)$. On the other hand, the characteristics due to the transport terms $U_{transport}(\cdot, \cdot)$ move across the boundary of the domain $[-3R, 3R] \times [-b_{max}, b_{max}]$ in the *wrong* direction (at least when $|b|$ is small): namely, boundary points move *inwards* when time decreases, which implies that the time evolution of h inside the domain cannot be separated from the evolution outside. A simpler way to understand this, looking by duality at the *reversed* characteristics for the associated Markov process Z , is to take a glance at Proposition 3.1 below: a terminal condition at time T of the form $x \mapsto \frac{1}{x-Z}$ with Z inside the domain gets transformed by the $(\frac{1}{x})$ -kernel into a function $\frac{C_t}{X-Z_t}$, where Z_t leaves the domain in finite time. This is all the more true when one incorporates the potential, since a point located on $\{\pm 3R\} \times [-b_{max}, b_{max}]$ moves out of the domain with a horizontal speed $\approx V'(\pm 3R)$ which can be arbitrarily large. This problem may be remedied at the cost of introducing horizontal and vertical *boundary terms*, which may be added to the remainder term in (2).
- (4) finally, one must check that the norm of the operators in (3.6) is bounded by some function of R which is compensated by the exponentially fast decreasing probability of the support not to be included in $[-R, R]$ (see (1)).

Let us collect by anticipation all the terms which will come out of the computations in §3.1 through §3.9. By explicit computation we show the following decomposition along the sum $h = h^{int} + h^{ext}$,

$$\mathcal{H}^\kappa = (\mathcal{H}_{transport}^\kappa \oplus 0) + \mathcal{H}_{nonlocal}^\kappa \quad (3.7)$$

namely, $\mathcal{H}^\kappa(h^{int} + h^{ext}) = \mathcal{H}_{transport}^\kappa h^{int} + \mathcal{H}_{nonlocal}^\kappa(h^{int} + h^{ext})$, with $\text{supp}(\mathcal{H}_{transport}^\kappa h^{int}) \subset [-3R, 3R] \times [-b_{max}, b_{max}]$, and correspondingly (taking the adjoint thereof)

$$\mathcal{L}^\kappa = (\mathcal{L}_{transport}^\kappa \oplus 0) + \mathcal{L}_{V=0}^{nonlocal}. \quad (3.8)$$

In turn,

$$\mathcal{H}_{transport}^\kappa := \mathcal{H}_0^\kappa + \sum_{k=0}^2 \mathcal{H}_{V-0}^{\kappa,(k)} \quad (3.9)$$

is a sum of 4 transport operators, which are *unbounded operators* $L^1(\Pi_{b_{max}}) \rightarrow L^1(\Pi_{b_{max}})$, and (using once again the decomposition along the sum $h = h^{int} + h^{ext}$)

$$\mathcal{H}_{nonlocal}^\kappa := \begin{pmatrix} \mathcal{H}_{nonlocal}^{\kappa,(int,int)} & \mathcal{H}_{nonlocal}^{\kappa,(ext,int)} \\ \mathcal{H}_{nonlocal}^{\kappa,(int,ext)} & \mathcal{H}_{nonlocal}^{\kappa,(ext,ext)} \end{pmatrix} = \begin{pmatrix} \mathcal{H}_{V-0}^{\kappa,(3),int} + \text{bdry} & \mathcal{H}_{V-0}^{\kappa,(ext,int)} \\ \mathcal{H}_{V-0}^{\kappa,(3),ext} + \text{bdry} & \mathcal{H}_{V-0}^{\kappa,(ext,ext)} \end{pmatrix}, \quad (3.10)$$

is a sum of *bounded operators* $L^1(\Omega \times [-b_{max}, b_{max}]) \rightarrow L^1(\Omega' \times [-b_{max}, b_{max}])$, $\Omega, \Omega' = [-3R, 3R]$ or $\mathbb{R} \setminus [-2R, 2R]$, with $\mathcal{H}_{V-0}^{\kappa,(3),int} = \bar{\chi}_R \mathcal{H}_{V-0}^{\kappa,(3)}$, $\mathcal{H}_{V-0}^{\kappa,(3),ext} = (1 - \bar{\chi}_R) \mathcal{H}_{V-0}^{\kappa,(3)}$, and similarly, $\mathcal{H}_{V-0}^{\kappa,(ext,int)} = \bar{\chi}_R \mathcal{H}_{V-0}^{\kappa,ext}$, $\mathcal{H}_{V-0}^{\kappa,(ext,ext)} = (1 - \bar{\chi}_R) \mathcal{H}_{V-0}^{\kappa,ext}$.

3.1 The $(\frac{1}{x})$ -kernel part

(1.33) is easily solved by the characteristic method in the case $V \equiv 0$ for test functions f of the form $f(x) = \frac{c}{x-z}$. Up to conjugation we may assume that $b := \text{Im } z > 0$. This is done in (Israelsson [13], Lemmas 2-4) – we need only subtract the trivial contribution of the harmonic potential – :

Proposition 3.1. (see Israelsson [13]) Assume $V \equiv 0$. Then eq. (1.33) with terminal condition $f_T(x) = \frac{c}{x-z}$ ($\text{Im } z > 0$) is solved as

$$f_t(x) = \frac{C_t}{x - Z_t}, \quad (3.11)$$

where $(C_t)_{0 \leq t \leq T}, (Z_t)_{0 \leq t \leq T}$ solve the following ode's,

$$\frac{dZ_t}{dt} = -\frac{\beta}{4}(M_t^N(Z_t) + M_t(Z_t)), \quad \frac{dC_t}{dt} = -\frac{\beta}{4}((M_t^N)'(Z_t) + M_t'(Z_t))C_t \quad (3.12)$$

with terminal conditions $z_T = z, c_T = c$. In particular,

$$\text{Im } \frac{dC_t}{dt} = -\frac{\beta}{4} \langle X_t^N + X_t, \text{Im } (f_{Z_t}) \rangle \leq 0. \quad (3.13)$$

Obviously, the solution of eq. (1.33) with terminal condition $f_T(x) = \frac{c}{x-z}$ is now $f_t(x) = \frac{\bar{C}_t}{x - \bar{Z}_t}$.

The last inequality (3.13), a simple consequence of (7.5), implies that z_t moves *away* from the real axis as t decreases, hence away from singularities. From the above and from general properties of the Stieltjes transform of ρ_t (see section 7), one can deduce bounds for the displacement, as in [13]. First $|M_t^N(z)|, |M_t(z)| \leq 1/b_t$, hence (letting $Z_t =: A_t + iB_t$, $B_t > 0$), for some large enough constant $C > 0$,

$$B_T \leq B_t \leq \sqrt{|B_T|^2 + C(T-t)}. \quad (3.14)$$

Similarly, $|A_t - A_T| = O\left(\frac{T-t}{B_T}\right)$. Finally, $\frac{\beta}{4} (|(M_t^N)'(Z_t)| + |(M_t)'(Z_t)|) \leq \frac{|dB_t/dt|}{B_t} = \left|\frac{d}{dt}(\ln(B_t))\right|$, whence $C_t \leq \frac{B_t}{B_T} = \sqrt{|B_T|^2 + C(T-t)} / B_T$. For a given initial condition z , $|\frac{dA_t}{dt}|, |\frac{dB_t}{dt}| \leq \frac{\beta}{2} \frac{1}{B_t} \leq \frac{\beta}{2} \frac{1}{b}$ is bounded along the characteristics.

Definition 3.2. Let $\mathcal{L}_0 = \mathcal{L}_0(t)$ be the time-dependent operator defined by

$$\begin{aligned} (\mathcal{L}_0(t)\phi)(a, b) = & -\frac{\beta}{4} \left(\operatorname{Re} [(M_t^N + M_t)(z)] \partial_a \phi(a, b) + \operatorname{Im} [(M_t^N + M_t)(z)] \partial_b \phi(a, b) \right. \\ & \left. + ((M_t^N)' + M_t')(z) \phi(a, b) \right) \end{aligned} \quad (3.15)$$

The motivation for this definition is the following. Let $f_t(z) = \frac{C_t}{x-Z_t}$ as in Proposition 3.1. Then $\frac{\partial f_t}{\partial t}|_{t=T} = (\mathcal{L}_0 f_T)(T; z) := (\mathcal{L}_0(T) f_T)(z)$. Take, more generally, a terminal condition for (1.33) of the form

$$f_T(x) = \int da_T \int db_T \frac{h_T(a_T, b_T)}{x - z_T}. \quad (3.16)$$

Then

$$\begin{aligned} \frac{\partial f_t}{\partial t}|_{t=T} &= \int da_T db_T (\mathcal{L}_0(\frac{1}{x-\cdot})(z_T) h_T(a_T, b_T)) \\ &= \int da_T db_T f_{z_T}(x) (\mathcal{L}_0^\dagger h_T)(T, z_T), \end{aligned} \quad (3.17)$$

where $(\mathcal{L}_0^\dagger h_T)(T; z_T) = ((\mathcal{L}_0(T))^\dagger h_T)(z_T)$ is obtained from the adjoint of $\mathcal{L}_0(T)$. Thus \mathcal{L}_0 , resp. \mathcal{L}_0^\dagger may be considered as the generator of a – here *deterministic* – *generalized* Markov process $Z = A + iB$, resp. the associated Fokker-Planck generator, where *generalized* refers to the supplementary order 0 term in $\mathcal{L}_0(t)$ (second line of (3.15)), which has on top of that the nasty property of not being even real-valued.

We now need some very general development, which we apply to \mathcal{L}_0 in this paragraph. Let $w : \Pi \rightarrow \mathbb{R}_+^*$ be some weight function. The relation $\frac{d}{dt}(wh) = w(\mathcal{L}_0^w(t))^\dagger h$ defines an operator $(\mathcal{L}_0^w(t))^\dagger$,

$$\begin{aligned} (\mathcal{L}_0^w)^\dagger(t; a, b) &:= (w \mathcal{L}_0(t) w^{-1})^\dagger(a, b) \\ &= -\frac{\beta}{4} w(a, b)^{-1} \left(-\partial_a \operatorname{Re} [(M_t^N + M_t)(z)] - \partial_b \operatorname{Im} [(M_t^N + M_t)(z)] \right. \\ &\quad \left. + ((M_t^N)' + M_t')(z) \right) w(a, b) \end{aligned} \quad (3.18)$$

which is the adjoint of \mathcal{L}_0 with respect to the measure $w(a, b) da db$ on Π . In other words, we see that the solution $(h_t)_{0 \leq t \leq T}$ of the equation

$$\frac{\partial h_t}{\partial t} = (\mathcal{L}_0^w)^\dagger(t) h_t \quad (3.19)$$

with terminal condition h_T is the *density* of Z . with respect to the measure $w(a, b) da db$.

Let us consider specifically the cases

$$w(a, b) := b |b|^\kappa, \quad \kappa = 0, 1, 2, \dots \quad (3.20)$$

for which we write

$$\mathcal{L}_0^w \equiv \mathcal{L}_0^\kappa. \quad (3.21)$$

These cases allow a direct connection to Stieltjes decompositions of order κ with $b_{max} = +\infty$, namely: if

$$f_T(x) = \mathcal{C}^\kappa h_T(x) = \int da \int_0^{+\infty} (-ib) db \frac{|b|^\kappa}{(1+\kappa)!} h_T(a, b) f_z(x) \quad (3.22)$$

then

$$f_t(x) := \mathcal{C}^\kappa h_t(x), \quad (3.23)$$

where $(h_t)_{0 \leq t \leq T}$ is the solution of

$$\frac{\partial h_t}{\partial t} = (\mathcal{L}_0^\kappa)^\dagger(t) h_t, \quad (3.24)$$

The generator

$$\mathcal{L}_0^\kappa = \left((b |b|^\kappa)^{-1} \mathcal{L}_0^\dagger b |b|^\kappa \right)^\dagger = |b|^{1+\kappa} \mathcal{L}_0 (|b|^{1+\kappa})^{-1} \quad (3.25)$$

is obtained from (3.15) by replacing the multiplicative term $\frac{\beta}{4}((M_t^N)' + M_t')(z)\phi(a, b)$ with $\frac{\beta}{4}(((M_t^N)' + M_t')(z) - \frac{1+\kappa}{b} \text{Im} [(M_t^N + M_t)(z)]) \phi(t; a, b)$, from which

$$\begin{aligned} \mathcal{H}_0^\kappa(h)(t; a, b) &:= (\mathcal{L}_0^\kappa)^\dagger(t)(h_t)(a, b) \\ &= \frac{\beta}{4} (\partial_a [\text{Re} ((M_t^N + M_t)(z)) h(t; a, b)] + \partial_b [\text{Im} ((M_t^N + M_t)(z)) h(t; a, b)] \\ &\quad + \left[\frac{1+\kappa}{b} \text{Im} ((M_t^N + M_t)(z)) - ((M_t^N)' + M_t')(z) \right] h(t; a, b) \end{aligned} \quad (3.26)$$

Consider the extended characteristics $(z_t, c_t) := (a_t + ib_t, c_t)$ of the operator \mathcal{H}_0^κ as in section 6: they are defined as the solution of

$$\frac{da_t}{dt} = \frac{\beta}{4} \text{Re} (M_t^N + M_t)(a_t + ib_t), \quad \frac{db_t}{dt} = \frac{\beta}{4} \text{Im} (M_t^N + M_t)(a_t + ib_t) \quad (3.27)$$

$$\frac{dc_t^\kappa}{dt} = \frac{\beta}{4} \left(\frac{1+\kappa}{b_t} \text{Im} [(M_t^N + M_t)(a_t + ib_t)] + ((\bar{M}_t^N)' + \bar{M}_t')(a_t + ib_t) \right) c_t^\kappa \quad (3.28)$$

Here we used the fact that $\partial_a \text{Re} (M_t^N + M_t)(z) = \partial_b \text{Im} (M_t^N + M_t)(z) = \text{Re} ((M_t^N)' + M_t')(z)$ since $z \mapsto (M_t^N + M_t)(z)$ is holomorphic. *Mind the sign changes* with respect to Proposition 3.1: characteristics of the dual operator \mathcal{H}_0^κ have reversed velocities with respect to those of \mathcal{L} (see section 6 for more details).

Now it follows from (7.14) that $\text{Re} (-\tau(t, z_t)) \equiv \text{Re} \left(-(c_t^\kappa)^{-1} \frac{dc_t^\kappa}{dt} \right) \leq 0$ for $\kappa \geq 0$, whence (see section 6) \mathcal{H}_0^κ is a generator of a semi-group of L^∞ -contractions. Because the first line of (3.26) is in divergence form, and the second line has positive real part, \mathcal{H}_0^κ is also the generator of a semi-group of L^1 -contractions.

Let us now see what happens for R and b_{max} *finite*. It is clear that the cut-off does not change the characteristic equations for $(a_t + ib_t, c_t)$. On the other hand, we get two supplementary *boundary term*. This can be proved as follows. Assume

$$f_T(x) = \mathcal{C}^\kappa h_T(x) = \int_{-3R}^{3R} da_T \int_{-b_{max}}^{b_{max}} (-ib_T) db_T \frac{|b|^\kappa}{(1+\kappa)!} h_T(a_T, b_T) f_{z_T}(x). \quad (3.29)$$

Then (coming back directly to the characteristics equations of Proposition 3.1)

$$\begin{aligned} & -\frac{\beta}{4} \int \int \frac{f'_T(x) - f'_T(y)}{x-y} (X_t^N(dy) + X_t(dy)) \\ &= -\frac{\beta}{4} \int_{-3R}^{+3R} da_T \int_{-b_{max}}^{b_{max}} (-ib_T) db_T \frac{|b_T|^\kappa}{(1+\kappa)!} \{ \text{Im} [(M_T^N + M_T)(z_T)] (\partial_{b_T} f_{z_T})(x) + \\ & \quad \text{Re} [(M_T^N + M_T)(z_T)] (\partial_{a_T} f_{z_T})(x) \} h_T(a_T, b_T) + \dots \\ &= \frac{\beta}{4} \int_{-3R}^{+3R} da_T \int_{-b_{max}}^{b_{max}} (-ib_T) db_T \frac{|b_T|^\kappa}{(1+\kappa)!} \{ \partial_{b_T} (\text{Im} [(M_T^N + M_T)(z_T)] h_T(a_T, b_T)) + \\ & \quad \partial_{a_T} (\text{Re} [(M_T^N + M_T)(z_T)] h_T(a_T, b_T)) \} f_{z_T}(x) + \dots \\ & \quad + \text{bdry} \end{aligned} \quad (3.30)$$

where "... " denote the contribution of the c -characteristics (which we can ignore), so (by integration by parts) we get a boundary term $\text{bdry} \equiv \text{h-bdry}_0 + \text{v-bdry}_0$ on the support of the measure, with

$$\begin{aligned} \text{h-bdry}_0 &= -\frac{\beta}{4} \frac{b_{max}^{1+\kappa}}{(1+\kappa)!} \int_{-\infty}^{+\infty} da_T \cdot \chi_R(x) \cdot (\text{Im} [(M_T^N + M_T)(a_T + ib_{max})]) \cdot \\ & \quad \cdot f_{a_T+ib_{max}}(x) h(a_T, b_{max}) - \text{Im} [(M_T^N + M_T)(a_T - ib_{max})] f_{a_T-ib_{max}}(x) h(a_T, -b_{max}) \end{aligned} \quad (3.31)$$

and

$$\begin{aligned} \text{v-bdry}_0 &= -\frac{\beta}{4} \int_{-b_{max}}^{b_{max}} (-ib_T) db_T \frac{|b_T|^\kappa}{(1+\kappa)!} \cdot \chi_R(x) \cdot \{ \text{Re} [(M_T^N + M_T)(3R + ib_T)] \cdot \\ & \quad \cdot f_{3R+ib_T}(x) h(3R, b_T) - \text{Re} [(M_T^N + M_T)(-3R + ib_T)] f_{-3R+ib_T}(x) h(-3R, b_T) \} \end{aligned} \quad (3.32)$$

3.2 The potential-dependent part: general introduction

Consider now the potential-dependent part in (1.33). *Without further mention we fix for the discussion $R \geq 1$ and some arbitrary $b_{max} \in (0, \frac{1}{2}]$.* Generally speaking we want to write the action of the operator $V'(x) \frac{\partial}{\partial x}$ on a function $f \equiv f_T$ with Stieltjes decomposition of order κ on $[-R, R]$

$$f(x) = \int_{-\infty}^{+\infty} da_T \int_{-b_{max}}^{b_{max}} (-ib_T) db_T \frac{|b_T|^\kappa}{(1+\kappa)!} f_{z_T}(x) h(a_T, b_T), \quad |x| \leq R \quad (3.33)$$

(see Definition 2.3) . In principle (see below though) it may be done in the following way.

Definition 3.3 (cut-offs). *Let $\chi_R : \mathbb{R} \rightarrow \mathbb{R}$ be a smooth cut-off function such that:*

$$(i) \quad \chi_R \equiv 1 \text{ on } [-R, R];$$

$$(ii) \quad \chi_R|_{B(0, \frac{3}{2}R)^c} \equiv 0;$$

$$(iii) \quad \forall k \geq 0, \|\chi_R^{(k)}\|_\infty = O(R^{-k})$$

and $\bar{\chi}_R$ be the function $x \mapsto \chi_R(\frac{x}{2})$.

To achieve condition (iii) it suffices to choose a C^∞ function χ_1 satisfying (i), (ii) and let $\chi_R(\cdot) := \chi_1(\frac{\cdot}{R})$.

Definition 3.4 (g -kernel). *Let, for $\kappa, \kappa' = 0, 1, 2, \dots$*

$$g^{\kappa'; \kappa}(a, b; a_T, b_T) := \mathbf{1}_{|b| < b_{max}} \frac{(-ib_T)|b_T|^\kappa}{(1 + \kappa)!} \mathcal{K}_{b_{max}}^{\kappa'} (x \mapsto \chi_R(x) V'(x) f'_{z_T}(x)) (a). \quad (3.34)$$

In practice we are only interested in couples of indices $(\kappa, \kappa' = \kappa)$ and $(\kappa, \kappa' = \kappa + 1)$, and let $g^\kappa \equiv g^{\kappa; \kappa}$. Let f be a function as in (3.33). Then, *using the standard order κ' Stieltjes decomposition with cut-off b_{max} of $V'(x) \frac{\partial}{\partial x}(f_{z_T}(x))$* , we get

$$\begin{aligned} V'(x) \frac{\partial}{\partial x} f(x) &= \int_{-\infty}^{+\infty} da \int_{-b_{max}}^{b_{max}} (-ib) db \frac{|b|^{\kappa'}}{(1 + \kappa')!} f_z(x) \\ &\quad \int_{-\infty}^{+\infty} da_T \int_{-b_{max}}^{b_{max}} db_T g^{\kappa'; \kappa}(a, b; a_T, b_T) h(a_T, b_T), \end{aligned} \quad (3.35)$$

thus defining (unbounded) operators $\mathcal{H}_{V-0}^\kappa, \mathcal{H}_{V-0}^{\kappa+1; \kappa} : L^1(\Pi_{b_{max}}, da_T db_T) \rightarrow L^1(\Pi_{b_{max}}, da db)$,

$$\begin{aligned} \mathcal{H}_{V-0}^\kappa(h)(a, b) &= \int_{-\infty}^{+\infty} da_T \int_{-b_{max}}^{b_{max}} db_T g^\kappa(a, b; a_T, b_T) h(a_T, b_T) \\ \mathcal{H}_{V-0}^{\kappa+1; \kappa}(h)(a, b) &= \int_{-\infty}^{+\infty} da_T \int_{-b_{max}}^{b_{max}} db_T g^{\kappa+1; \kappa}(a, b; a_T, b_T) h(a_T, b_T). \end{aligned}$$

However, these Stieltjes representations of the vector field $V'(x) \frac{\partial}{\partial x}$ will be used directly only when $h = 0 \oplus h^{ext}$ has support in $\mathbb{R} \setminus [-2R, 2R]$, producing the \mathcal{H}^{ext} -term. When $h = h^{int} \oplus 0$ has support $\subset [-3R, 3R]$, we separate first the Taylor expansion of order 2 of $V'(x)$ around a_T , as explained in (1.12) or (3.37) below, which is directly analyzed without further Stieltjes decomposition.

Let us discuss in details how we proceed when $h = h^{int}$. As we have just mentioned, the first step is to use a second-order Fourier-expansion of V' around a_T for $|a_T| \leq 3R$ and $x \in \text{supp}(\chi_R)$, i.e. $|x| \leq \frac{3}{2}R$,

$$V'(x) = V'(a_T) + V''(a_T)(x - a_T) + V'''(a_T) \frac{(x - a_T)^2}{2} + (x - a_T)^3 W_{a_T}(x - a_T), \quad (3.36)$$

where W_{a_T} is C^7 . Thus

$$V'(x)\frac{\partial}{\partial x} = V'(a_T)\partial_x + V''(a_T)(x - a_T)\partial_x + V'''(a_T)\frac{(x - a_T)^2}{2}\partial_x + (x - a_T)^3W_{a_T}(x - a_T)\partial_x \quad (3.37)$$

makes four different contributions to the generator, $\mathcal{H}_{V-0}^{(k)}$, $k = 0, 1, 2, 3$, resp. called *constant, linear, quadratic and Taylor remainder term*, plus some boundary contributions. Computations show the following: $\mathcal{H}_{V-0}^{(k)}$, $k = 0, 1, 2$, are directly of the adjoint form

$$h \mapsto \left((a_T, b_T) \mapsto \partial_{a_T}(v_{hor}(a_T, b_T)h(a_T, b_T)) + \partial_{b_T}(v_{vert}(a_T, b_T)h(a_T, b_T)) - \tau(a_T, b_T)h(a_T, b_T) \right) \quad (3.38)$$

with $\text{Re } \tau(\cdot) \leq 0$, implying that *they generate L^1 -contractions* (see section 6, and recall that we go *backwards* in time). As a nice feature of this problem, $\mathcal{H}_{V-0}^{(k)}$, $k = 0, 1, 2$ *also generate L^∞ -contractions* (but wait!). Replacing the vector field $\chi_R(x)V'(x)\partial_x$ in (3.34) above by $\chi_R(x)(x - a_T)^3W_{a_T}(x - a_T)\partial_x$ produces a kernel $g_{V-0}^{(3)}(a, b; a_T, b_T)$ discussed in §3.8. As for the first three terms, they are directly shown to be equivalent to the action of a transport operator (see §3.4, 3.5, 3.6). We sum up in §3.7 the contributions of the transport operators introduced in §3.1, 3.4, 3.5 and 3.6. Finally, the contribution of the boundary terms is analyzed in §3.9.

Unfortunately, the terms in $\mathcal{H}_{nonlocal}$ *do not* generate neither L^∞ - nor L^1 -contractions. Thanks to the horizontal and vertical cut-offs however, they are bounded, hence generate by integration some exponentially increasing time factor $e^{C_R t}$, with C_R depending on R . The R -dependence of the constant C_R must be carefully evaluated, since $e^{C_R t}$ must be in the end counterbalanced by our large deviation estimates for the support.

3.3 Away from the support

We study in this subsection the g -kernels $g_{V-0}^{\kappa'; \kappa, ext}$ obtained by assuming that $h = h^{ext}$, whence $|a_T| \geq 2R$. We want to prove that the operators

$$\mathcal{H}_{V-0}^{\kappa, ext} : h \mapsto \left(\mathcal{H}^{\kappa, ext}(h) : (a, b) \mapsto \int da_T \int_{-b_{max}}^{b_{max}} db_T g_{V-0}^{\kappa; \kappa, ext}(a, b; a_T, b_T) h(a_T, b_T) \right) \quad (3.39)$$

and

$$\mathcal{H}_{V-0}^{\kappa+1; \kappa, ext} : h \mapsto \left(\mathcal{H}^{\kappa+1; \kappa, ext}(h) : (a, b) \mapsto \int da_T \int_{-b_{max}}^{b_{max}} db_T g_{V-0}^{\kappa+1; \kappa, ext}(a, b; a_T, b_T) h(a_T, b_T) \right) \quad (3.40)$$

are bounded operators in $L^1(\Pi_{b_{max}})$. We shall actually only prove the statement for $\mathcal{H}_{V-0}^{\kappa+1; \kappa, ext}$, and leave the similar proof of the statement for $\mathcal{H}_{V-0}^{\kappa, ext}$ to the reader. First we introduce scaled norms:

Definition 3.5. For $f \in C^k(\mathbb{R}, \mathbb{R})$ and $R > 0$, let

$$\|f\|_{k, [-R, R]} := \sum_{j=0}^k R^j \sup_{[-R, R]} |f^{(j)}|. \quad (3.41)$$

Note that, if $f = P$ is a polynomial of order d , then $\|P\|_{k,[-R,R]} = O(R^d)$ for all k when R is large. Also, by assumption, $\|\chi_R\|_{k,[-\frac{3}{2}R,\frac{3}{2}R]} = O(1)$.

Lemma 3.6. (i) $\|\mathcal{H}^{\kappa+1;\kappa,ext}\|_{L^1(\Pi_{b_{max}}) \rightarrow L^1(\Pi_{b_{max}})} = O\left(b_{max}^{-1}\|V'\|_{4+\kappa,[-\frac{3}{2}R,\frac{3}{2}R]}R^{-1}\right);$

(ii) $\|\mathcal{H}^{\kappa,ext}\|_{L^1(\Pi_{b_{max}}) \rightarrow L^1(\Pi_{b_{max}})} = O\left(\|V'\|_{3+\kappa,[-\frac{3}{2}R,\frac{3}{2}R]}R^{-1}\right).$

Remark. L^∞ -operator norms $\|\cdot\|_{L^\infty(\Pi_{b_{max}}) \rightarrow L^\infty(\Pi_{b_{max}})}$ for $\mathcal{H}^{\kappa';\kappa,ext}$ ($\kappa' = \kappa, \kappa + 1$) satisfy the same estimates. As can be checked by looking at the details of the proof, norms $\|\cdot\|_{L^1(\Pi_{b_{max}}) \rightarrow L^\infty(\Pi_{b_{max}})}$ are deduced from these by dividing by a volume factor $\text{Vol}([-3R, 3R] \times [-b_{max}, b_{max}]) \approx Rb_{max}$, which follows from the fact that the kernel $g_{V-0}^{\kappa';\kappa,ext}$ is regular on the diagonal.

Proof.

(1) (operator norm of $\mathcal{H}_{V-0}^{\kappa+1;\kappa,ext}$) By definition,

$$g_{V-0}^{\kappa+1;\kappa,ext}(a, b; a_T, b_T) = \mathbf{1}_{|b| < b_{max}} \mathbf{1}_{|a_T| \geq 2R} \frac{1}{(1+\kappa)!} (-ib) |b_T|^\kappa \mathcal{K}_{b_{max}}^{\kappa+1}(\chi_R V' f'_{z_T})(a) \quad (3.42)$$

Note that, since $|a_T| \geq 2R$ and $\text{supp}(\chi_R) \subset B(0, \frac{3}{2}R)$, the function $\chi_R V' f'_{z_T}$ is regular and bounded.

We consider in the computations only the part g^κ of the kernel $g_{V-0}^{\kappa+1;\kappa,ext}$ obtained by replacing \mathcal{K}^κ with the operators in factor of the bounded operator $1 + \underline{\mathcal{K}}_{b_{max}}^\kappa$ in (2.13, 2.16), and distinguish the cases κ odd, κ even. Assume first κ is *odd*. Then

$$\begin{aligned} & |g^\kappa(a, b; a_T, b_T)| \\ &= \mathbf{1}_{|b| < b_{max}} \mathbf{1}_{|a_T| \geq 2R} O(|b_T|^{1+\kappa}) \left(|(\chi_R V' f'_{z_T})^{(3+\kappa)}(a)| + b_{max}^{-(3+\kappa)} |(\chi_R V' f'_{z_T})(a)| \right) \\ &= O\left(R^{-(3+\kappa)} b_{max}^{1+\kappa} \|V'\|_{3+\kappa,[-\frac{3}{2}R,\frac{3}{2}R]} + b_{max}^{-2} \|V'\|_{0,[-\frac{3}{2}R,\frac{3}{2}R]} \right) \\ &\quad R^{-2} \mathbf{1}_{(a,b) \in [-3R, 3R] \times [-b_{max}, b_{max}]}. \end{aligned} \quad (3.43)$$

independently of a_T, b_T . Hence $\mathcal{H}_{V-0}^{\kappa+1;\kappa,ext}$ is a bounded operator on $L^1(\Pi_{b_{max}})$, with L^1 -operator norm

$$\begin{aligned} \|\mathcal{H}_{V-0}^{\kappa+1;\kappa,ext}\| &= O\left(R^{-(3+\kappa)} b_{max}^{1+\kappa} \|V'\|_{3+\kappa,[-\frac{3}{2}R,\frac{3}{2}R]} + |b|_{max}^{-2} \|V'\|_{0,[-\frac{3}{2}R,\frac{3}{2}R]}\right) R^{-2} \cdot \\ &\quad \cdot \text{Vol}([-3R, 3R] \times [-b_{max}, b_{max}]) \\ &= O\left(\left(\frac{b_{max}}{R}\right)^{3+\kappa} \|V'\|_{3+\kappa,[-\frac{3}{2}R,\frac{3}{2}R]} + \|V'\|_{0,[-\frac{3}{2}R,\frac{3}{2}R]}\right) b_{max}^{-1} R^{-1}. \end{aligned} \quad (3.44)$$

Assume now that κ is *even*. Then (using (3.43) and (7.23))

$|g^\kappa(a, b; a_T, b_T)| \leq O(b_{max}^{-2} \|V'\|_{0, [-\frac{3}{2}R, \frac{3}{2}R]} R^{-2} \mathbf{1}_{(a,b) \in [-3R, 3R] \times [-b_{max}, b_{max}]})$, plus

$$\begin{aligned} & \mathbf{1}_{|b| < b_{max}} O(|b_T|^{1+\kappa}) \left| \mathbf{1}_{|a_T| \geq 2R} p.v. \left(\frac{1}{x} \right) * (\chi_R V' f'_{z_T})^{(3+\kappa)}(a) \right| \\ &= \mathbf{1}_{|b| < b_{max}} O(|b_T|^{1+\kappa}) \left| \mathbf{1}_{|a_T| \geq 2R} p.v. \left(\frac{1}{x^2} \right) * (\chi_R V' f'_{z_T})^{(2+\kappa)}(a) \right| \\ &= O(|b_T|^{1+\kappa}) \left[\mathbf{1}_{(a,b) \in [-3R, 3R] \times [-b_{max}, b_{max}]} R \|\mathbf{1}_{|a_T| \geq 2R} (\chi_R V' f'_{z_T})^{(4+\kappa)}\|_\infty \right. \\ & \quad \left. + \mathbf{1}_{(a,b) \in (\mathbb{R} \setminus [-3R, 3R]) \times [-b_{max}, b_{max}]} \frac{R}{a^2} \|\mathbf{1}_{|a_T| \geq 2R} (\chi_R V' f'_{z_T})^{(2+\kappa)}\|_\infty \right]. \end{aligned} \quad (3.45)$$

We conclude by multiplying by $\bar{\chi}_R + (1 - \bar{\chi}_R)$ that $\mathcal{H}_{V-0}^{\kappa+1; \kappa, ext} h = \tilde{h}^{int} + \tilde{h}^{ext}$, where

$$\|\tilde{h}^{int}\|_{L^1} \leq O\left(\left(\frac{b_{max}}{R}\right)^{3+\kappa} \|V'\|_{4+\kappa, [-\frac{3}{2}R, \frac{3}{2}R]} + \|V'\|_{0, [-\frac{3}{2}R, \frac{3}{2}R]}\right) b_{max}^{-1} R^{-1} \|h\|_{L^1} \quad (3.46)$$

and

$$\|\tilde{h}^{ext}\|_{L^1} \leq O\left(\left(\frac{b_{max}}{R}\right)^{2+\kappa} \|V'\|_{2+\kappa, [-\frac{3}{2}R, \frac{3}{2}R]} R^{-2}\right) \|h\|_{L^1}. \quad (3.47)$$

(2) (operator norm of $\mathcal{H}_{V-0}^{\kappa, ext}$) With respect to (3.44), (3.46) and (3.47), we remove one order of differentiation and one power of b_{max}^{-1} , and exchange parities. Thus

$$|||\mathcal{H}_{V-0}^{\kappa, ext}||| = O\left(\left(\frac{b_{max}}{R}\right)^{2+\kappa} \|V'\|_{2+\kappa, [-\frac{3}{2}R, \frac{3}{2}R]} + \|V'\|_{0, [-\frac{3}{2}R, \frac{3}{2}R]}\right) R^{-1} \quad (3.48)$$

for κ even, while for κ odd, $\mathcal{H}_{V-0}^{\kappa, ext} h = \tilde{h}^{int} + \tilde{h}^{ext}$, with

$$\|\tilde{h}^{int}\|_{L^1} \leq O\left(\left(\frac{b_{max}}{R}\right)^{2+\kappa} \|V'\|_{3+\kappa, [-\frac{3}{2}R, \frac{3}{2}R]} + \|V'\|_{0, [-\frac{3}{2}R, \frac{3}{2}R]}\right) R^{-1} \|h\|_{L^1} \quad (3.49)$$

and

$$\|\tilde{h}^{ext}\|_{L^1} \leq O\left(\left(\frac{b_{max}}{R}\right)^{2+\kappa} \|V'\|_{1+\kappa, [-\frac{3}{2}R, \frac{3}{2}R]} R^{-1}\right) \|h\|_{L^1}. \quad (3.50)$$

Remark. When $b_{max} = +\infty$, the adjoint of $\mathcal{H}_{V-0}^{\kappa, ext}$ or $\mathcal{H}^{\kappa+1; \kappa, ext}$ is the generator of a *signed jump Markov process* with good properties. Namely, for all a_T, b_T ,

$$\int_{-\infty}^{+\infty} da \int_{-\infty}^{+\infty} db g_{V-0}^{\kappa, ext}(a, b; a_T, b_T) = 0 \quad (3.51)$$

since $K_{+\infty}^\kappa(s=0) = 0$, hence $\mathcal{L}_{V-0}^{\kappa, ext} := \mathcal{H}^{\kappa, ext}$ may be written in the following form,

$$\mathcal{L}_{V-0}^{\kappa, ext}(\phi)(a_T, b_T) = \int_{-\infty}^{+\infty} da \int_{-\infty}^{+\infty} db (\phi(a, b) - \phi(a_T, b_T)) g_{V-0}^{\kappa, ext}(a, b; a_T, b_T). \quad (3.52)$$

For a bona fide Markov process, the function $(a, b) \mapsto -g^{\kappa, ext}(a, b; a_T, b_T) \geq 0$ would be the jump rate from (a_T, b_T) to (a, b) , and one would have $-\int da db g^{\kappa, ext}(a, b; a_T, b_T) = 1$. Here g is signed, so the probabilistic interpretation fails stricto sensu. However, the L^1 semi-group generated by $\mathcal{H}_{V-0}^{\kappa, ext}$ has good properties because $\mathcal{H}_{V-0}^{\kappa, ext}$ is a bounded operator (see section 4).

3.4 Constant term

Inserting the constant operator $V'(a_T) \frac{\partial}{\partial x}$ inside the Stieltjes decomposition

$$f_T(x) = \int_{-3R}^{+3R} da_T \int_{-b_{max}}^{b_{max}} (-ib_T) db_T |b_T|^\kappa f_{z_T}(x) h(a_T, b_T),$$

we get

$$\begin{aligned} & \int_{-3R}^{+3R} da_T \int_{-b_{max}}^{b_{max}} (-ib_T) db_T |b_T|^\kappa V'(a_T) f'_{z_T}(x) h(a_T, b_T) \\ &= - \int_{-3R}^{+3R} da_T \int_{-b_{max}}^{b_{max}} (-ib_T) db_T |b_T|^\kappa V'(a_T) \frac{\partial}{\partial a_T} (f_{z_T}(x)) h(a_T, b_T) \\ &= \int_{-3R}^{+3R} da_T \int_{-b_{max}}^{b_{max}} (-ib_T) db_T |b_T|^\kappa f_{z_T}(x) \frac{\partial}{\partial a_T} (V'(a_T) h(a_T, b_T)) + \text{bdry}. \end{aligned} \tag{3.53}$$

where

$$\begin{aligned} \text{bdry} &\equiv \text{v-bdry}_{V-0}^{(0)} = - \int_{-b_{max}}^{b_{max}} (-ib_T) db_T |b_T|^\kappa \cdot \chi_R(x) \cdot \\ &\cdot (V'(3R) f_{3R+ib_T}(x) h(3R, b_T) - V'(-3R) f_{-3R+ib_T}(x) h(-3R, b_T)) \end{aligned} \tag{3.54}$$

is a vertical boundary term coming from integration by parts, which we shall not discuss till §3.9.

This defines a new operator $\mathcal{H}_{V-0}^{(0)}$ in divergence form,

$$\mathcal{H}_{V-0}^{\kappa, (0)}(h)(a_T, b_T) = \frac{\partial}{\partial a_T} (V'(a_T) h(a_T, b_T)), \tag{3.55}$$

with corresponding extended characteristics

$$\frac{da_t}{dt} = V'(a_t), \quad \frac{db_t}{dt} = 0, \quad \frac{dc_t^\kappa}{dt} = V''(a_t). \tag{3.56}$$

3.5 Linear term

Proceeding as in the *constant term* case, we insert the operator $V''(a_T)(x - a_T)\frac{\partial}{\partial x}$ inside the Stieltjes decomposition of f_T , getting

$$\begin{aligned} & \int da_T \int_{-b_{max}}^{b_{max}} (-ib_T) db_T |b_T|^\kappa V''(a_T) [(x - a_T)\partial_x f_{z_T}(x)] h(a_T, b_T) \\ &= - \int da_T \int_{-b_{max}}^{b_{max}} (-ib_T) db_T |b_T|^\kappa V''(a_T) \cdot \partial_{b_T}(b_T f_{z_T}(x)) h(a_T, b_T) \\ &= \text{bdry} + \int da_T \int_{-b_{max}}^{b_{max}} (-ib_T) db_T |b_T|^\kappa f_{z_T}(x) V''(a_T) (b_T \partial_{b_T} + 1 + \kappa) h(a_T, b_T), \end{aligned} \quad (3.57)$$

where

$$\begin{aligned} \text{bdry} \equiv \text{bdry}_{V-0}^{(1)} &= -b_{max}^{2+\kappa} \int da_T V''(a_T) \cdot \chi_R(x) \cdot \\ &\cdot (f_{a_T+ib_{max}}(x) h(a_T, b_{max}) - f_{a_T-ib_{max}}(x) h(a_T, -b_{max})) \end{aligned} \quad (3.58)$$

is a horizontal boundary term coming from integration by parts. This defines an operator, $\mathcal{H}_{V-0}^{\kappa,(1)}$,

$$\mathcal{H}_{V-0}^{\kappa,(1)}(h)(a_T, b_T) = (\partial_{b_T} b_T + \kappa) (V''(a_T) h(a_T, b_T)). \quad (3.59)$$

with associated characteristics,

$$\frac{da_t}{dt} = 0, \quad \frac{db_t}{dt} = V''(a_t) b_t, \quad \frac{dc_t^\kappa}{dt} = (1 + \kappa) V''(a_t) c_t^\kappa \quad (3.60)$$

3.6 Quadratic term

Proceeding as in the previous paragraph, we insert the operator $\frac{1}{2}V'''(a_T)(x - a_T)^2\frac{\partial}{\partial x}$ inside the Stieltjes decomposition. Since $(x - a_T)^2\frac{\partial}{\partial x} = \partial_x(x - a_T)^2 - 2(x - a_T)$, and

$$\partial_x [(x - a_T)^2 f_{z_T}(x)] = 1 + b_T^2 \partial_{a_T} f_{z_T}(x), \quad -2(x - a_T) f_{z_T}(x) = -2 - 2ib_T f_{z_T}(x) \quad (3.61)$$

this term produces a new operator

$$\mathcal{H}_{V-0}^{\kappa,(2)}(h)(a_T, b_T) = \frac{1}{2} (-\partial_{a_T} b_T^2 - 2ib_T) (V'''(a_T) h(a_T, b_T)). \quad (3.62)$$

with associated characteristics

$$\frac{da_t}{dt} = -\frac{1}{2} V'''(a_t) b_t^2, \quad \frac{db_t}{dt} = 0, \quad \frac{dc_t^\kappa}{dt} = -i V'''(a_t) b_t c_t^\kappa \quad (3.63)$$

plus a vertical boundary term,

$$\begin{aligned} \text{bdry} \equiv \text{v} - \text{bdry}_{V-0}^{(2)} &= - \int_{-b_{max}}^{b_{max}} (-ib_T) db_T |b_T|^{\kappa+2} \cdot \chi_R(x) \cdot \\ &\cdot (V'''(3R) f_{3R+ib_T}(x) h(3R, b_T) - V'''(-3R) f_{-3R+ib_T}(x) h(-3R, b_T)) \end{aligned} \quad (3.64)$$

The remaining term due to the constant $\frac{1}{2}V'''(a_T)(1 - 2)$ in (3.61), integrated with respect to the measure $|b_T|^{1+\kappa} da_T db_T$, adds a time-independent constant C to f_t , which however disappears from the computations since: (i) the right-hand side of (1.33) features only f'_t ; (ii) $d\langle Y_t^N, C \rangle = dC = 0$ in (1.8).

3.7 Recapitulating: the transport contribution

We can now write down the action of the sum of our three transport operators,

$$\mathcal{H}_{transport}^\kappa(t) := \mathcal{H}_0^\kappa(t) + \mathcal{H}_{V-0}^{\kappa,(0)} + \mathcal{H}_{V-0}^{\kappa,(1)} + \mathcal{H}_{V-0}^{\kappa,(2)} \quad (3.65)$$

(note that only \mathcal{H}_0^κ depends on the time variable), defined respectively in (3.26), (3.55), (3.59) and (3.62). Summing the contributions in (3.28), (3.56), (3.60) and (3.63), we obtain the following equation for the characteristics on $[-3R, 3R] \times [-b_{max}, b_{max}]$,

$$\frac{da_t}{dt} = \frac{\beta}{4} \operatorname{Re} (M_t^N + M_t)(a_t + ib_t) + V'(a_t) - \frac{1}{2} V'''(a_t) b_t^2 \quad (3.66)$$

$$\frac{db_t}{dt} = \frac{\beta}{4} \operatorname{Im} (M_t^N + M_t)(a_t + ib_t) + V''(a_t) b_t \quad (3.67)$$

$$\begin{aligned} \frac{dc_t^\kappa}{dt} = & \left[\frac{\beta}{4} \left(\frac{1+\kappa}{b_t} \operatorname{Im} (M_t^N + M_t)(a_t + ib_t) + ((\bar{M}_t^N)') + (\bar{M}_t)')(a_t + ib_t) \right) \right. \\ & \left. + (2+\kappa)V''(a_t) - iV'''(a_t)b_t \right] c_t^\kappa. \end{aligned} \quad (3.68)$$

Let us make the following observations (see §3.1):

- (i) $t \mapsto |b_t|$ increases, whence $|b_t| \leq |b_T|$ ($0 \leq t \leq T$). By using the same trick as in §3.1, i.e. by computing $\frac{d(b_t^2)}{dt}$, one sees that b -characteristics go to 0 in finite time. Unfortunately this is a priori true in general even if a_T is large, beyond the support, for instance, if $|a_T| = 3R$: if $V'(\cdot)$ has a superlinear increase, then (for large R) time-reversed characteristics issued from (a_T, b_T) at time T move to (a_t, b_t) with $|a_T| = R$ in very short time, after which the only available bound for the vertical velocity due to the $(\frac{1}{x})$ -potential is $\approx 1/b_t$; in particular, the exponentially decreasing factor $e^{-b/\rho}$ in the function $h(a, b) := e^{-b/\rho} \mathcal{K}_{b_{max}, \rho}^0(\operatorname{Im} f_{z_T})$ in (2.29) is lost very quickly under the time evolution, inq;
- (ii) as is already true of each individual generator \mathcal{H}_0^κ , $\mathcal{H}_{V-0}^{\kappa,(i)}$ ($i = 0, 1, 2$), the sum $\mathcal{H}_{transport}^\kappa(t)$ generates a semi-group of L^1 -contractions. The same holds if one considers L^∞ -norms, since the real part of (3.68) is positive.

Section 6 therefore implies:

Lemma 3.7. *Let $u_T \in L^1([-3R, 3R] \times (0, b_{max}]) \cap C^1([-3R, 3R] \times (0, b_{max}])$ and $\kappa = 0, 1, 2, \dots$. Then the backward evolution equation $\frac{du}{dt} = \mathcal{H}_{transport}^\kappa(t)u(t)$, $u|_{t=T} = u_T$ ($0 \leq t \leq T$) has a unique solution $u(t) := U_{transport}(t, T)u_T$, such that*

$$\|u_t\|_{L^1([-3R, 3R] \times (0, b_{max}])} \leq \|u_T\|_{L^1([-3R, 3R] \times (0, b_{max}])} \quad (3.69)$$

$$\|u_t\|_{L^\infty([-3R, 3R] \times (0, b_{max}])} \leq \|u_T\|_{L^\infty([-3R, 3R] \times (0, b_{max}])} \quad (3.70)$$

In section 4, we will separate (3.68) into its real and imaginary parts. Solving the (a, b) -characteristics coupled with the real characteristic \tilde{c}^κ ,

$$\frac{d\tilde{c}_t^\kappa}{dt} = \text{Re} \left[\frac{\beta}{4} \left(\frac{1+\kappa}{b_t} \text{Im} (M_t^N + M_t)(a_t + ib_t) + ((\bar{M}_t^N)' + (\bar{M}_t)')(a_t + ib_t) \right) + (2 + \kappa)V''(a_t) - iV'''(a_t)b_t \right] \tilde{c}_t^\kappa, \quad (3.71)$$

one gets a backward evolution equation, $\frac{d\tilde{u}}{dt} = [\text{Re } \mathcal{H}_{\text{transport}}^\kappa(t)]\tilde{u}(t)$, $\tilde{u}|_{t=T} = \tilde{u}_T \equiv u_T$ ($0 \leq t \leq T$), solved as $\tilde{u}(t) := \tilde{U}_{\text{transport}}(t, T)\tilde{u}_T$, such that

$$\|\tilde{u}_t\|_{L^1([-3R, 3R] \times (0, b_{\max}))} \leq \|\tilde{u}_T\|_{L^1([-3R, 3R] \times (0, b_{\max}))} \quad (3.72)$$

$$\|\tilde{u}_t\|_{L^\infty([-3R, 3R] \times (0, b_{\max}))} \leq \|\tilde{u}_T\|_{L^\infty([-3R, 3R] \times (0, b_{\max}))} \quad (3.73)$$

Assuming $u_T \geq 0$, \tilde{u} is simply the modulus of u ,

$$\tilde{u}_t(a, b) = |u_t(a, b)|. \quad (3.74)$$

In particular, $\tilde{u}_t \geq 0$. The *adjoint evolution* with generator $(\text{Re } \mathcal{H}_{\text{transport}}^\kappa)^\dagger$ is *sub-Markovian*, i.e. \tilde{u}_t is the density at time t of a (deterministic) time-reversed Markov $(\bar{A}_t, \bar{B}_t)_{0 \leq t \leq T}$ process with kernel $p(t, \bar{a}_t, \bar{b}_t; s, \bar{a}_s, \bar{b}_s) \geq 0$ ($t \leq s$) such that $\int d\bar{a} \int d\bar{b} p(t, \bar{a}, \bar{b}; s, \bar{a}_s, \bar{b}_s) \leq 1$. Note that bars \bar{a}, \bar{b} are here to recall that trajectories (\bar{A}_t, \bar{B}_t) differ from the above characteristics (a_t, b_t) – velocities are reversed for the adjoint evolution (note that we did not use bars in §3.1, to avoid possible confusion with complex conjugation).

Remark. It is instructive to look at terms that would be produced by continuing the Taylor expansion to infinity. Note that the boundary value of the operator $\mathcal{H}_{V=0}^{(2)}(h)$ vanishes ($\mathcal{H}_{V=0}^{(2)}(h)(a_T, b_T = 0) \equiv 0$ vanishes to order ≥ 1). It may be proven in general that the contribution of order j vanishes to order $\geq j - 1$. Summing up the whole series would yield (up to κ -dependent terms)

$$\frac{da_t}{dt} = \sum_{n=2p} \frac{V^{(n+1)}(a_t)}{n!} (ib_t)^n, \quad \frac{db_t}{dt} = -i \sum_{n=2p+1} \frac{V^{(n+1)}(a_t)}{n!} (ib_t)^n \quad (3.75)$$

$$\frac{dc_t}{dt} = \sum_{n \geq 1} V^{(n+1)}(a_t) \frac{(ib_t)^{n-1}}{(n-1)!}. \quad (3.76)$$

Note that, for V *holomorphic*, this is equivalent to the vector field $V'(Z_t)\partial_{Z_t} + V''(Z_t)$. However, terms of order ≥ 3 also produce *polynomials* in x (instead of linear combination of $f_z(x)$, $z \in \mathbb{C} \setminus \mathbb{R}$). Integrating the generator would yield power series in x which are maybe controllable, but this would require totally different techniques with respect to ours.

3.8 Main remainder term

We study in this subsection the g -kernels $g_{V=0}^{\kappa'; \kappa, (3)}$ obtained by replacing $\chi_R(x)V'(x)f'_{z_T}(x)$ with $\chi_R(x)(x-a_T)^3 W_{a_T}(x-a_T)f'_{z_T}(x)$ in Definition 3.4. We want to prove that the operators

$$\mathcal{H}^{\kappa, (3)} : h \mapsto \mathcal{H}^{\kappa, (3)}(h) : \left((a, b) \mapsto \int_{-3R}^{3R} da_T \int_{-b_{\max}}^{b_{\max}} db_T g_{V=0}^{\kappa; \kappa, (3)}(a, b; a_T, b_T) h(a_T, b_T) \right) \quad (3.77)$$

and

$$\mathcal{H}^{\kappa+1;\kappa,(3)} : h \mapsto \mathcal{H}^{\kappa+1;\kappa,(3)}(h) : \left((a, b) \mapsto \int_{-3R}^{3R} da_T \int_{-b_{max}}^{b_{max}} db_T g_{V-0}^{\kappa+1;\kappa,(3)}(a, b; a_T, b_T) h(a_T, b_T) \right) \quad (3.78)$$

are bounded operators in $L^1(\Pi_{b_{max}})$. Our result is as in §3.3, apart from the fact that one needs to control more derivatives of V :

Lemma 3.8. (i) $|||\mathcal{H}^{\kappa+1;\kappa,(3)}|||_{L^1(\Pi_{b_{max}}) \rightarrow L^1(\Pi_{b_{max}})} = O(b_{max}^{-1} \|V'\|_{8+\kappa, [-3R, 3R]} R^{-1});$

(ii) $|||\mathcal{H}^{\kappa,(3)}|||_{L^1(\Pi_{b_{max}}) \rightarrow L^1(\Pi_{b_{max}})} = O(\|V'\|_{7+\kappa, [-3R, 3R]} R^{-1}).$

As in §3.3, we obtain the same estimates for the L^∞ -operator norms $||| \cdot |||_{L^\infty(\Pi_{b_{max}}) \rightarrow L^\infty(\Pi_{b_{max}})}$.

We shall actually only prove the statement for $\mathcal{H}^{\kappa+1;\kappa,(3)}$. The proof of (ii) reduces immediately to (i) by substituting $\kappa \rightarrow \kappa - 1$ and taking into account the extra b -prefactor.

Remark. Had we Taylor expanded V' to order 2 instead of order 3, would we have obtained an *unbounded* operator instead of $\mathcal{H}^{\kappa+1;\kappa,(3)}$, as testified by the bound (3.88) below (with x^2 instead of x^3 in the numerator, the integral would diverge in the limit $b_T \rightarrow 0$ for the maximum value of m). On the other hand, for the same reason, it is easy to see by looking at the details of the proof that $\mathcal{H}^{\kappa,(3)} : L^1(\Pi_{b_{max}}) \rightarrow L^\infty(\Pi_{b_{max}})$ is bounded, typically because in (3.88) one obtains instead of the L^1 -kernel $x \mapsto \frac{b_T}{x^2 + b_T^2}$ the bounded function $x \mapsto \frac{b_T^2}{x^2 + b_T^2}$.

Proof. By definition,

$$g_{V-0}^{\kappa+1;\kappa,(3)}(a, b; a_T, b_T) = \frac{(-ib_T)|b_T|^\kappa}{(1+\kappa)!} \mathcal{K}_{b_{max}}^{\kappa+1} \left(x \mapsto \chi_R(x)(x - a_T)^3 W_{a_T}(x - a_T) f'_{z_T}(x) \right) (a)$$

Proceeding as in §3.3, we consider in the computations only the part g^κ of the kernel $g_{V-0}^{\kappa+1;\kappa,(3)}$ obtained by replacing \mathcal{K}^κ with the operators in factor of the bounded operator $1 + \underline{\mathcal{K}}_{b_{max}}^\kappa$ in (2.13, 2.16). For some numerical constants $c_1 = c_1(\kappa)$, $c_2 = c_2(\kappa)$,

$$\begin{aligned} g_{V-0}^\kappa(a, b; a_T, b_T) &= (-ib_T)|b_T|^\kappa \left\{ c_1 [\mathcal{F}^{-1}(|s|^{3+\kappa}) *] + c_2 b_{max}^{-(3+\kappa)} \right\} (F_{a_T+ib_T})(x) \\ &=: (-ib_T)|b_T|^\kappa (c_1 G_{a_T+ib_T}^1(x) + c_2 G_{a_T+ib_T}^2(x)), \end{aligned} \quad (3.79)$$

with $x := a - a_T$ and

$$F_{a_T+ib_T}(y) := \tilde{W}_{a_T}(y) \frac{y^3}{(y - ib_T)^2} \quad (3.80)$$

where $\tilde{W}_{a_T}(y) = \chi_R(a_T + y) W_{a_T}(y)$ has support $\subset [-\frac{9}{2}R, \frac{9}{2}R]$ since $|a_T| \leq 3R$. Thus (3.77) looks like a convolution formula in the coordinates a, a_T – but not quite since \tilde{W}_{a_T} depends on a_T –, which leads us to use the following bound (where $G_{a_T+ib_T} := c_1 G_{a_T+ib_T}^1 + c_2 G_{a_T+ib_T}^2$)

$$\begin{aligned} |||\mathcal{H}^{\kappa+1;\kappa,(3)}(h)|||_{L^1(\Pi_{b_{max}})} &\leq b_{max} \left(\sup_{(a_T, b_T) \in [-3R, 3R] \times [-b_{max}, b_{max}]} |b_T|^{1+\kappa} \right. \\ &\quad \left. \int da |G_{a_T+ib_T}(a - a_T)| \right) |||h|||_{L^1(\Pi_{b_{max}})} \end{aligned} \quad (3.81)$$

whence

$$|||\mathcal{H}^{\kappa+1;\kappa,(3)}||| \leq b_{max} \left(\sup_{(a_T, b_T) \in [-3R, 3R] \times [-b_{max}, b_{max}]} |||b_T|^{1+\kappa} G_{a_T+ib_T}|||_{L^1} \right) ||h||_{L^1(\Pi_{b_{max}})}. \quad (3.82)$$

We must therefore bound $|||b_T|^{1+\kappa} G_{a_T+ib_T}|||_{L^1}$. *The main issue, on which we shall now concentrate, is to bound $|||b_T|^{1+\kappa} G_{a_T+ib_T}^1|||_{L^1([-5R, 5R])}$; as shown later on, the missing terms are less singular and may be bounded similarly.*

If κ is *odd* then $|s|^{3+\kappa} = s^{3+\kappa}$ is the Fourier symbol of a differential operator, otherwise $|s|^{3+\kappa} = \text{sgn}(s)s^{3+\kappa}$ involves a further convolution by a singular kernel. Let us accordingly distinguish two cases. But first of all we let, for $\ell \geq 0$ and $|a_T| \leq 3R$,

$$C_\ell := \sum_{\ell'=3}^{\ell} R^{-(\ell-\ell')} ||\tilde{W}_{a_T}^{(\ell'-3)}||_{\infty} \quad (3.83)$$

and note that

$$C_\ell = O \left(\sum_{\ell'=3}^{\ell} R^{-(\ell-\ell')} \sup_{[-3R, 3R]} |V^{(\ell'+1)}| \right) = O(R^{-\ell} ||V'||_{\ell, [-3R, 3R]}) \quad (3.84)$$

and $\ell \mapsto R^\ell C_\ell$ is increasing. More generally, if $\ell \geq \ell' \geq \ell'' \geq 3$ and $r \geq 0$,

$$||y \mapsto (\tilde{W}_{a_T}^{(\ell''-3)}(y)y^r)^{(\ell'-\ell'')}||_{\infty} = O(R^{\ell-\ell'+r} C_\ell). \quad (3.85)$$

(i) (κ odd) First

$$|||b_T|^{1+\kappa} G_{a_T+ib_T}^1|||_{L^1(\mathbb{R})} = O(C_{6+\kappa} |b_T|^{1+\kappa}) \sum_{m=0}^{3+\kappa} \int_{-5R}^{5R} dx R^m \left| \partial_x^m \left(\frac{x^3}{(x-ib_T)^2} \right) \right|. \quad (3.86)$$

Then

$$\partial_x^m \left(\frac{x^3}{(x-ib_T)^2} \right) = \sum_{p-q=1-m, p \leq 3, q \geq 2} C_{p,q}^m x^p (x-ib_T)^{-q} \quad (3.87)$$

for some coefficients $C_{p,q}^m$. If $3 \leq m \leq 3+\kappa$ then $|x^p (x-ib_T)^{-q}| = O(\frac{|x-ib_T|^{3-m}}{x^2+|b_T|^2}) = O(\frac{|b_T|^{3-m}}{x^2+|b_T|^2})$. Multiplying with respect to $|b_T|^{1+\kappa}$ and integrating, one gets (using $b_T \leq b_{max} \leq 1 \leq R$)

$$\begin{aligned} R^m ||b_T|^{1+\kappa} \partial_x^m \left(\frac{x^3}{(x-ib_T)^2} \right) |||_{L^1(\mathbb{R})} &\leq C' R^m b_T^{3+\kappa-m} \int dx \frac{b_T}{x^2 + |b_T|^2} \\ &\leq R^{3+\kappa}. \end{aligned} \quad (3.88)$$

If $m \leq 2$ then $b_T \left| \partial_x^m \left(\frac{x^3}{(x-ib_T)^2} \right) \right| = O(b_T \frac{|x|^{3-m}}{x^2+|b_T|^2}) = O(|x|^{2-m})$. Thus $R^m |||b_T|^{1+\kappa} \partial_x^m \left(\frac{x^3}{(x-ib_T)^2} \right) |||_{L^1([-5R, 5R])} = O(|b_T|^\kappa R^3) = O(b_{max}^\kappa R^3)$.

All together:

$$|||b_T|^{1+\kappa} G_{a_T+ib_T}^1|||_{L^1(\mathbb{R})} = O(C_{6+\kappa} R^{3+\kappa}). \quad (3.89)$$

(ii) (κ even) Let, for $0 \leq m \leq 3 + \kappa$,

$$I(x) := p.v. \left(\frac{1}{x} \right) * \left[x \mapsto \tilde{W}_{a_T}^{(3+\kappa-m)}(x) \partial_x^m \left(\frac{x^3}{(x - ib_T)^2} \right) \right]. \quad (3.90)$$

We must bound $|b_T|^{1+\kappa} \int dx |I(x)|$.

We rewrite $I(x)$ as the sum of two contributions (we refer to section 7 without further mention for computations and bounds related to the principal value integral), $I(x) =: I_{reg}(x) + I_{sing}(x)$, where

$$I_{reg}(x) := \int dy \frac{\tilde{W}_{a_T}^{(3+\kappa-m)}(x) - \tilde{W}_{a_T}^{(3+\kappa-m)}(y)}{x - y} \partial_x^m \left(\frac{x^3}{(x - ib_T)^2} \right) \quad (3.91)$$

and

$$I_{sing}(x) := \int dy \tilde{W}^{(3+\kappa-m)}(y) \frac{\partial_x^m \left(\frac{x^3}{(x - ib_T)^2} \right) - \partial_y^m \left(\frac{y^3}{(y - ib_T)^2} \right)}{x - y}. \quad (3.92)$$

Using $\left| \frac{\tilde{W}_{a_T}^{(3+\kappa-m)}(x) - \tilde{W}_{a_T}^{(3+\kappa-m)}(y)}{x - y} \right| \leq \|\tilde{W}_{a_T}^{(4+\kappa-m)}\|_\infty \leq R^m C_{7+\kappa}$, and noting that, for $|x| \leq 5R$, the integral $\int dy (\dots) = \int_{B(x, \frac{19}{2}R)} (\dots)$ (by symmetry) simply produces an extra factor $O(R)$, one obtains, using (i)

$$|b_T|^{1+\kappa} \int dx |I_{reg}(x)| = O(C_{7+\kappa} R^{4+\kappa}). \quad (3.93)$$

Considering now $I_{sing}(x)$, we expand the numerator of (3.92) as in (3.87) and rewrite

$$\begin{aligned} \frac{x^p(x - ib_T)^{-q} - y^p(y - ib_T)^{-q}}{x - y} &= \frac{x^p - y^p}{x - y} (y - ib_T)^{-q} + x^p \frac{(y - ib_T)^q - (x - ib_T)^q}{(x - y)(x - ib_T)^q(y - ib_T)^q} \\ &= \sum_{r=0}^{p-1} x^{p-1-r} \frac{y^r}{(y - ib_T)^q} - \sum_{r'=1}^q \frac{x^p}{(x - ib_T)^{r'}} \frac{1}{(y - ib_T)^{q+1-r'}}. \end{aligned} \quad (3.94)$$

The integrals $J_1 := \int dy \frac{\tilde{W}_{a_T}^{(3+\kappa-m)}(y)y^r}{(y - ib_T)^q}$, $J_2 := \int dy \frac{\tilde{W}_{a_T}^{(3+\kappa-m)}(y)}{(y - ib_T)^{q+1-r'}}$ may be bounded using (7.19) since $q \geq 2 \geq 1$ and $q + 1 - r' \geq 1$. Thus

$$|J_1| = O(C_{8+\kappa} R^{4-p+r}), |J_2| = O(C_{8+\kappa} R^{3+r'-p}) \quad (3.95)$$

Then $\int_{-5R}^{5R} dx |x|^{p-1-r} = O(R^{p-r})$ and

$$|b_T|^{1+\kappa} \int dx \frac{|x|^p}{|x - ib_T|^{r'}} = O(|b_T|^{(1+\kappa)-(r'-p-1)}) = O(b_{max}^{2-r'+\kappa+p}) \quad (r' \geq p+2) \quad (3.96)$$

(note that $2 - r' + \kappa + p \geq 0$);

$$|b_T|^{1+\kappa} \int_{-5R}^{5R} dx \frac{|x|^p}{|x - ib_T|^{r'}} \leq |b_T|^{1+\kappa} \int_{-5R}^{5R} \frac{dx}{|x - ib_T|} = O(b_{max}^{1+\kappa} \ln(R/b_{max})) \quad (r' = p+1) \quad (3.97)$$

$$|b_T|^{1+\kappa} \int_{-5R}^{5R} dx \frac{|x|^p}{|x - ib_T|^{r'}} \leq b_{max}^{1+\kappa} \int_{-5R}^{5R} dx |x|^{p-r'} = O(b_{max}^{1+\kappa} R^{p-r'+1}) \quad (r' \leq p) \quad (3.98)$$

All together, using $b_{max} \leq R$, one obtains, summing all terms:

$$|b_T|^{1+\kappa} \int dx |I_{sing}(x)| = O(C_{8+\kappa} R^{5+\kappa}). \quad (3.99)$$

Let us now quickly deal with the missing terms. First

$$\begin{aligned} |||b_T|^{1+\kappa} G_{a_T+ib_T}^2|||_{L^1} &\leq b_{max}^{-2} \int dx \frac{|x|^3}{|x - ib_T|^2} |\tilde{W}_{a_T}(x)| \\ &\leq b_{max}^{-2} |||\tilde{W}_{a_T}|||_{\infty} \int_{-\frac{9}{2}R}^{\frac{9}{2}R} dx |x| = O\left(\left(\frac{b_{max}}{R}\right)^{-2} C_3\right). \end{aligned} \quad (3.100)$$

Then one must bound $|b_T|^{1+\kappa} \int_{|x| \geq 5R} dx |G_{a_T+ib_T}^1(x)|$; because $\text{supp}(\tilde{W}_{a_T}) \subset [-\frac{9}{2}R, \frac{9}{2}R]$, this contribution vanishes except if κ is *even*, see (ii) above, in which case (by integration by parts)

$$\begin{aligned} |b_T|^{1+\kappa} \int_{|x| \geq 5R} dx |G_{a_T+ib_T}^1(x)| &= |b_T|^{1+\kappa} \int_{|x| \geq 5R} dx \left| \int_{-\frac{9}{2}R}^{\frac{9}{2}R} dy p.v. \left(\frac{1}{(x-y)^2} \right) F_{a_T+ib_T}^{(2+\kappa)}(y) \right| \\ &\leq b_T \int_{|x| \geq 5R} dx O\left(\frac{1}{x^2}\right) |||b_T|^\kappa F_{a_T+ib_T}^{(2+\kappa)}|||_{L^1([-5R, 5R])}. \end{aligned} \quad (3.101)$$

The integral $|||b_T|^\kappa F_{a_T+ib_T}^{(2+\kappa)}|||_{L^1([-5R, 5R])}$ is (up to the replacement $\kappa \rightarrow \kappa - 1$) exactly the one which has been computed in case (i) above. Hence we find:

$$|b_T|^{1+\kappa} \int_{|x| \geq 5R} dx |G_{a_T+ib_T}^1(x)| = O\left(\left(\frac{b_T}{R}\right) C_{5+\kappa} R^{2+\kappa}\right). \quad (3.102)$$

Since $C_3 \leq R^{2+\kappa} C_{5+\kappa} \leq R^{3+\kappa} C_{6+\kappa} \leq R^{5+\kappa} C_{8+\kappa} = O(R^{-3} |||V'|||_{8+\kappa, [-3R, 3R]})$ and $\frac{b_{max}}{R} \leq 1$, the sum of estimates (3.89, 3.93, 3.99, 3.100, 3.102) is $O\left(\left(\frac{b_{max}}{R}\right)^{-2} R^{-3} |||V'|||_{[-3R, 3R], 8+\kappa}\right) = O(b_{max}^{-2} R^{-1} |||V'|||_{[-3R, 3R], 8+\kappa})$. \square

3.9 Boundary term

Recall the *horizontal boundary terms* h-bdry₀ (3.31), h-bdry _{V_{-0}} ⁽¹⁾ (3.58), and the *vertical boundary terms*, v-bdry₀ (3.32), v-bdry _{V_{-0}} ⁽⁰⁾ (3.54) and v-bdry _{V_{-0}} ⁽²⁾ (3.64). We adopt the following notations. First let

$$\partial_R \Pi_{b_{max}} := \partial([-3R, 3R] \times [-b_{max}, b_{max}]) = ([-3R, 3R] \times \{\pm b_{max}\}) \cup (\{\pm 3R\} \times [-b_{max}, b_{max}]), \quad (3.103)$$

and, for $h : [-3R, 3R] \times [-b_{max}, b_{max}] \rightarrow \mathbb{C}$, let $\partial_R h := h|_{\partial_R \Pi_{b_{max}}}$ be the restriction of h to the boundary. Horizontal boundary terms h-bdry \cdot are of the form h-bdry $\cdot(b_{max}; \partial_R h)$ – h-bdry $\cdot(-b_{max}; \partial_R h)$; similarly, vertical boundary terms v-bdry \cdot are of the form v-bdry $\cdot(3R; \partial_R h)$ – v-bdry $\cdot(-3R; \partial_R h)$.

These terms introduce a priori distribution-valued test functions h with support on $\partial_R \Pi_{b_{max}}$. In order to avoid that, we replace $f_{a_T \pm i b_{max}}$ with $\mathcal{C}^{\kappa'} \left((a, b) \mapsto \mathcal{K}_{b_{max}}^{\kappa'} (f_{a_T \pm i b_{max}})(a) \right)$, $\kappa' = \kappa, \kappa + 1$ so that

$$\begin{aligned} & (\text{h-bdry}_0 + \text{h-bdry}_{V-0}^{(1)})(b_{max}; \partial_R h) - (b_{max} \leftrightarrow -b_{max}) \\ &= \mathcal{C}^{\kappa+1}(\mathcal{H}^{\kappa+1; \kappa, h\text{-bdry}}(\partial_R h)) = \mathcal{C}^{\kappa}(\mathcal{H}^{\kappa, h\text{-bdry}}(\partial_R h)) \end{aligned} \quad (3.104)$$

with $\mathcal{H}^{\kappa+1; \kappa, h\text{-bdry}}, \mathcal{H}^{\kappa, h\text{-bdry}} : L^\infty(\partial_R \Pi_{b_{max}}) \rightarrow (L^1 \cap L^\infty)(\Pi_{b_{max}})$,

$$\mathcal{H}^{\kappa+1; \kappa, h\text{-bdry}}(\partial_R h)(a, b) = \int_{-3R}^{3R} da_T g_{b_{max}}^{\kappa+1; \kappa, h\text{-bdry}}(a, b; a_T) \partial_R h(a_T, b_{max}) - (b_{max} \leftrightarrow -b_{max}) \quad (3.105)$$

$$\mathcal{H}^{\kappa, h\text{-bdry}}(\partial_R h)(a, b) = \int_{-3R}^{3R} da_T g^{\kappa, h\text{-bdry}}(a, b; a_T) \partial_R h(a_T, b_{max}) - (b_{max} \leftrightarrow -b_{max}) \quad (3.106)$$

and

$$\begin{aligned} g_{b_{max}}^{\kappa'; \kappa, h\text{-bdry}}(a, b; a_T) &= - \left\{ \frac{\beta}{4} \frac{b_{max}^{1+\kappa}}{(1+\kappa)!} \text{Im} [(M_T^N + M_T)(a_T + i b_{max})] \right. \\ &\quad \left. + b_{max}^{2+\kappa} V''(a_T) \right\} \mathcal{K}_{b_{max}}^{\kappa'}(f_{a_T + i b_{max}})(a) \end{aligned} \quad (3.107)$$

Similarly, we replace $f_{\pm 3R + i b}$ with $\mathcal{C}^{\kappa'} \left((a, b) \mapsto \mathcal{K}_{b_{max}}^{\kappa'} (f_{\pm 3R + i b})(a) \right)$, $\kappa' = \kappa, \kappa + 1$, so that

$$\begin{aligned} & (\text{v-bdry}_0 + \text{v-bdry}_{V-0}^{(0)} + \text{v-bdry}_{V-0}^{(2)})(3R; \partial_R h) - (R \leftrightarrow -R) \\ &= \mathcal{C}^{\kappa+1}(\mathcal{H}^{\kappa+1; \kappa, v\text{-bdry}}(\partial_R h)) = \mathcal{C}^{\kappa}(\mathcal{H}^{\kappa, v\text{-bdry}}(\partial_R h)) \end{aligned} \quad (3.108)$$

with $\mathcal{H}^{\kappa+1; \kappa, v\text{-bdry}}, \mathcal{H}^{\kappa, v\text{-bdry}} : L^\infty(\partial_R \Pi_{b_{max}}) \rightarrow (L^1 \cap L^\infty)(\Pi_{b_{max}})$,

$$\mathcal{H}^{\kappa+1; \kappa, v\text{-bdry}}(\partial_R h)(a, b) = \int_{-b_{max}}^{b_{max}} db_T g_R^{\kappa+1; \kappa, v\text{-bdry}}(a, b; b_T) \partial_R h(3R, b_T) - (R \leftrightarrow -R) \quad (3.109)$$

$$\mathcal{H}^{\kappa, v\text{-bdry}}(\partial_R h)(a, b) = \int_{-b_{max}}^{b_{max}} db_T g_R^{\kappa, v\text{-bdry}}(a, b; b_T) \partial_R h(3R, b_T) - (R \leftrightarrow -R) \quad (3.110)$$

and

$$\begin{aligned} g_R^{\kappa'; \kappa, v\text{-bdry}}(a, b; b_T) &= -(-i b_T) \left\{ \frac{\beta}{4} \frac{|b_T|^\kappa}{(1+\kappa)!} \text{Re} [(M_T^N + M_T)(3R + i b_T)] \right. \\ &\quad \left. + (|b_T|^\kappa V'(3R) + |b_T|^{\kappa+2} V'''(3R)) \right\} \mathcal{K}_{b_{max}}^{\kappa'}(\chi_{|R|} f_{3R + i b_T})(a) \end{aligned} \quad (3.111)$$

Lemma 3.9. *Let $\kappa \geq 0$ and $\kappa' = \kappa, \kappa + 1$. Then*

$$(i) \quad |||\mathcal{H}^{\kappa'; \kappa, h-bdry}|||_{L^\infty(\partial_R \Pi_{b_{max}}) \rightarrow L^\infty(\Pi_{b_{max}})} = b_{max}^{-2-(\kappa'-\kappa)} \left\{ O\left(\frac{1}{b_{max}}\right) + O(\|V''\|_{0,[-3R,3R]}) \right\} \text{ and} \\ |||\mathcal{H}^{\kappa'; \kappa, h-bdry}|||_{L^\infty(\partial_R \Pi_{b_{max}}) \rightarrow L^1(\Pi_{b_{max}})} = R b_{max}^{-1-(\kappa'-\kappa)} \left\{ O\left(\frac{1}{b_{max}}\right) + O(\|V''\|_{0,[-3R,3R]}) \right\};$$

(ii)

$$|||\mathcal{H}^{\kappa'; \kappa, v-bdry}|||_{L^\infty(\partial_R \Pi_{b_{max}}) \rightarrow L^\infty(\Pi_{b_{max}})} \\ = R^{-1} b_{max}^{-1-(\kappa'-\kappa)} \left\{ O\left(\frac{1}{b_{max}}\right) + O(\|V'\|_{0,[-3R,3R]}) + O(b_{max}^2 \|V'''\|_{0,[-3R,3R]}) \right\} \quad (3.112)$$

and

$$|||\mathcal{H}^{\kappa, v-bdry}|||_{L^\infty(\partial_R \Pi_{b_{max}}) \rightarrow L^1(\Pi_{b_{max}})} \\ = b_{max}^{-(\kappa'-\kappa)} \left\{ O\left(\frac{1}{b_{max}}\right) + O(\|V'\|_{0,[-3R,3R]}) + O(b_{max}^2 \|V'''\|_{0,[-3R,3R]}) \right\}. \quad (3.113)$$

As in §3.3, L^1 -estimates and L^∞ -estimates differ only by a volume factor $\approx R b_{max}$.

Proof.

- (i) Immediate consequence of the bounds $|\text{Im} [(M_T^N + M_T)(a_T + i b_{max})]| \leq 2/b_{max}$, $|V''(a_T)| \leq \|V''\|_{0,[-3R,3R]}$ and $|\mathcal{K}_{b_{max}}^{\kappa'}(f_{a_T \pm i b_{max}})(a)| = O(b_{max}^{-(3+\kappa')}) O(\frac{1}{1+|a|/R^2})$ (see (2.12)).
- (ii) Immediate consequence of the bounds $|\text{Re} [(M_T^N + M_T)(a_T + i b_{max})]| \leq 2/b_{max}$, $|V'(3R)| \leq \|V'\|_{0,[-3R,3R]}$, $b_T^2 |V'''(3R)| \leq b_{max}^2 \|V'''\|_{0,[-3R,3R]}$, and $|\mathcal{K}_{b_{max}}^{\kappa'}(\chi_R f_{\pm 3R + i b_T})(a)| = O(\frac{b_{max}^{-(2+\kappa')}}{R}) O(\frac{1}{1+|a|/R^2})$ (see (2.12)).

□

4 Gaussianity of the fluctuation process

In this section, we prove our Main Theorem (see Introduction), namely, we prove that the finite- N fluctuation process $(Y_t^N)_{t \geq 0}$ converges weakly in $C([0, T], H_{-8})$ to a fluctuation process $(Y_t)_{t \geq 0}$, which is the unique solution of a martingale problem that we solve explicitly in terms of the solution of (1.33).

We fix once and for all: $b_{max} = \frac{1}{2}$ (see discussion in §1.3). Though this choice is not necessarily optimal (depending on V), it simplifies the discussion because all factors $|b|^\alpha$, $\alpha \in \mathbb{R}$ in the estimates are $O(1)$.

Summarizing what we have found up to now and applying Proposition 1.3, we find for $\kappa = 0, 1, 2, \dots$:

$$d\langle Y_t^N, \mathcal{C}^\kappa h_t \rangle = \frac{1}{2} \left(1 - \frac{\beta}{2}\right) \langle X_t^N, (\mathcal{C}^\kappa h_t)'' \rangle dt + \frac{1}{\sqrt{N}} \sum_i (\mathcal{C}^\kappa h_t)'(\lambda_t^i) dW_t^i \quad (4.1)$$

where $(h_t)_{t \leq T}$ is the solution of the evolution equation

$$\frac{dh_t}{dt} = \mathcal{H}^\kappa(t)h(t). \quad (4.2)$$

On the other hand, the process $(Y_t^N)_{t \geq 0}$ is a solution of the following *martingale problem*: if $\bar{\phi} = \{\phi_j\}_{1 \leq j \leq k}$ is a family of test functions in $C_c^\infty(\mathbb{R}, \mathbb{R})$, $F \in C_b^2(\mathbb{R}^k, \mathbb{R})$, then, letting $F_{\bar{\phi}}(Y^N) := F(\langle Y^N, \phi_1 \rangle, \dots, \langle Y^N, \phi_k \rangle)$,

$$\Phi_t^{T,N}(Y^N) := F_{\bar{\phi}}(Y_T^N) - F_{\bar{\phi}}(Y_t^N) - \int_t^T ds L_s^N F_{\bar{\phi}}(Y_s^N) \quad (4.3)$$

is a martingale, where

$$\begin{aligned} L_s^N F_{\bar{\phi}}(Y_s^N) := & \sum_{j=1}^k \frac{\partial F_{\bar{\phi}}}{\partial x_j}(Y_s^N) \left(\langle Y_s^N, \frac{\beta}{4} \int \frac{\phi'_j(\cdot) - \phi'_j(y)}{\cdot - y} (X_t + X_t^N)(dy) - V'(\cdot) \phi'_j(\cdot) \right) + \\ & + \frac{1}{2} (1 - \frac{\beta}{2}) \langle X_t^N, \phi_j'' \rangle + \frac{1}{2} \sum_{j,l=1}^k \partial_{jl}^2 F_{\bar{\phi}}(Y_s^N) \langle X_t^N, \phi_j' \phi_l' \rangle \end{aligned} \quad (4.4)$$

(see [13], p. 28–29).

We now use an exponential functional of the process to derive the limit law.

Definition 4.1. For $\kappa = 0, 1, 2, \dots$ and $h \in (L^1 \cap L^\infty)(\Pi_{b_{max}})$, let

$$\phi_h(Y_t^N) := e^{i \langle Y_t^N, \mathcal{C}^0 h \rangle}. \quad (4.5)$$

Itô's formula implies as in ([13], p. 29)

$$d\phi_{h_t}(Y_t^N) = \phi_{h_t}(Y_t^N) \left(\frac{1}{2} i (1 - \frac{\beta}{2}) \langle X_t^N, (\mathcal{C}^0 h_t)'' \rangle - \frac{1}{2} \langle X_t^N, ((\mathcal{C}^0 h_t)')^2 \rangle \right) dt, \quad (4.6)$$

from which for $0 \leq t \leq T$ (letting *formally* $N \rightarrow \infty$)

$$\mathbb{E}[\phi_{h_T}(Y_T) | \mathcal{F}_t] = \phi_{h_t}(Y_t) \mathbb{E} \left[\exp \left(\frac{1}{2} \int_t^T \left[i (1 - \frac{\beta}{2}) \langle X_s, (\mathcal{C}^0 h_s)'' \rangle - \langle X_s, ((\mathcal{C}^0 h_s)')^2 \rangle \right] ds \right) \right] \quad (4.7)$$

Since h_s , $t \leq s \leq T$ are linear in h_T , the term in the exponential in (4.7) is a sum of a linear and a quadratic term in h_T , giving resp. the expectation and the variance of a Gaussian process (see Israelsson, §2.6 for more details).

The strategy of the proof, following closely the proof in (Israelsson [13], section 2), is the following:

- (A) find bounds for $\mathbb{E}[\sup_{s \leq T} |\langle Y_s^N, \phi \rangle|]$, $\mathbb{E}[\sup_{s \leq T} |\langle Y_s^N, \int \frac{\phi'(\cdot) - \phi'(y)}{\cdot - y} X_s^N(dy) \rangle|]$ and $\mathbb{E}[\sup_{s \leq T} |\langle Y_s^N, V'(\cdot) \phi'(\cdot) \rangle|]$ (see first line in (1.35) or (4.4));
- (B) prove a *tightness property* for the family of processes Y^N , implying the existence of a (non necessarily unique) limit in law;

(C) prove that *any* weak limit Y of the $(Y^N)_{n \geq 1}$ satisfies the limit martingale problem, i.e. is such that

$$\Phi_t^T(Y) = F_{\bar{\phi}}(Y_T) - F_{\bar{\phi}}(Y_t) - \int_t^T ds L_s F_{\bar{\phi}}(Y_s) \quad (4.8)$$

is a martingale, where

$$\begin{aligned} L_s F_{\bar{\phi}}(Y_s) := & \sum_{j=1}^k \frac{\partial F_{\bar{\phi}}}{\partial x_j}(Y_s) \left(\langle Y_s, \frac{\beta}{2} \int \frac{\phi'_j(\cdot) - \phi'_j(y)}{\cdot - y} X_t(dy) - V'(\cdot) \phi'_j(\cdot) \rangle + \right. \\ & \left. + \frac{1}{2} (1 - \frac{\beta}{2}) \langle X_t, \phi''_j \rangle \right) + \frac{1}{2} \sum_{j,l=1}^k \partial_{jl}^2 F_{\bar{\phi}}(Y_s) \langle X_t, \phi'_j \phi'_l \rangle \end{aligned} \quad (4.9)$$

(obtained formally from (4.4) by letting $N \rightarrow \infty$);

(D) prove that there exists only *one* measure with given initial measure satisfying (4.9), and that it is Gaussian, and satisfies (4.7).

To prove (C) one must also prove that $(\Phi_t^T - \Phi_t^{T,N})(Y^N)$ is small.

We do not intend to rewrite Israelsson's paper, so we only emphasize the differences. First (C) is proved as in [13], Lemma 20 provided one can prove the following *bound*,

$$\mathbb{E}[\sup_{s \leq T} |\langle Y_s^N, \phi \rangle|] \leq C_T \|\phi\|_{H_6}, \quad (4.10)$$

from which one gets bounds for

$$\mathbb{E}[\sup_{s \leq T} |\langle Y_s^N, \int \frac{\phi'(\cdot) - \phi'(y)}{\cdot - y} X_s^N(dy) \rangle|], \mathbb{E}[\sup_{s \leq T} |\langle Y_s^N, V'(\cdot) \phi'(\cdot) \rangle|]. \quad (4.11)$$

The above H_6 -bound (4.10) is obtained using the integral formula (1.35), namely,

$$\begin{aligned} \mathbb{E}[\sup_{s \leq T} |\langle Y_s^N, \phi \rangle|] \leq & C (\mathbb{E}[|\langle Y_0^N, \phi \rangle|] + \\ & + \int_0^T ds \mathbb{E}[|\langle Y_s^N, V'(\cdot) \phi'(\cdot) \rangle| + |\langle Y_s^N, \int \frac{\phi'(x) - \phi'(\cdot)}{x - \cdot} (X_s^N(dx) + X_s(dx)) \rangle|] \\ & + \int_0^T ds \mathbb{E}[\int \phi''(x) X_s^N(dx)] + \mathbb{E}[\sup_{s \leq T} |M_s|]) \end{aligned} \quad (4.12)$$

where $M_s := \frac{1}{\sqrt{N}} \int_0^s \sum_{i=1}^N \phi'(\lambda_s^i) dW_s^i$ is a martingale. All these terms are bounded as in [13], Lemma 15, using easy, V -independent inequalities *and* the following *fundamental estimate*,

Lemma 4.2. *There exists a constant C depending on T such that, for all $t \leq T$,*

$$\mathbb{E}[|\langle Y_t^N, \phi \rangle|^2] \leq C \|\phi\|_{H_4}^2. \quad (4.13)$$

Note the loss of regularity with respect to ([13], Lemma 14), where one has $\|\phi\|_{H_2}$ in the r.-h.s. (this loss is due to the necessity of controlling both Y_0^N and the support, see (4.27)). This changes the bounds in the course of the proof of ([13], Lemma 15), where estimates are proved using Israelsson's Lemma 8 for $q = 2$ (with $q - 1$ playing the same rôle as our κ), yielding (with our notations) a bound on $\|\mathcal{K}_{b_{max}}^\kappa(f)\|_{L^1}$ in terms of $\|f\|_{L^1} + \|f^{(\kappa+2)}\|_{L^1}$, resp. $\|f\|_{L^1} + \|f^{(\kappa+3)}\|_{L^1}$, depending whether κ is even, resp. odd; because of the loss of regularity one must take $q = 4$, which increases the Sobolev index by two. Also, the term $\mathbb{E}[|U_s^2|^2]$ in Lemma 15, here $U_s^2 = \langle Y_s^N, V' \phi' \rangle$, can be controlled by $\|\phi\|_{H^5}^2$ just like $\mathbb{E}[|\langle Y_s^N, \phi' \rangle|^2]$, by a trivial adaptation of the proof of Lemma 4.2, using our large deviation estimates for the support of Y_s^N .

Then the tightness property (B) is proved using a lemma due to Mitoma [21] and the above estimates (see [13], §2.4); the Sobolev space H_{-8} in our Main Theorem has been chosen in such a way that there exists a nuclear mapping $H_8 \rightarrow H_6$ (see 4.10), as follows from Treves [29], which is a requirement in Mitoma's hypotheses.

Finally, letting $\phi_i = \mathcal{C}^0 h_i$, $i = 1, \dots, k$, (D) is proved by computing $\mathbb{E}[\exp i(\langle Y_{t_1}, \phi_1 \rangle + \dots + \langle Y_{t_k}, \phi_k \rangle)]$, $0 \leq t_k \leq \dots \leq t_1 \leq T$ by induction on k using the assumed martingale property of the limit(s) and solving in terms of the time-evolved functions $h_i(t)$, $i = 1, \dots, k$. For $k = 2$ we obtain (4.7).

So everything boils down to the proof of the above lemma.

Proof of Lemma 4.2. Let $\phi \in C_c^\infty$. Consider its standard Stieltjes decomposition of order 2, $\phi = \mathcal{C}^2 h$ (take $b_{max} = \frac{1}{2}$). Then

$$|\langle Y_t^N, \phi \rangle|^2 \leq \|h\|_{L^2(\Pi_{b_{max}})}^2 \int_{\Pi_{b_{max}}} da db b^6 (N|M_t(z) - M_t^N(z)|)^2 \quad (4.14)$$

and $\|h\|_{L^2(\Pi_{b_{max}})} = O(\|\phi\|_{H^4})$ (use estimates for the kernel $\mathcal{K}_{b_{max}}^1$ proved in section 2, together with Parseval-Bessel's formula). Hence (4.13) follows if we can show that

$$\mathbb{E}[|N(M_T^N(z) - M_T(z))|^2] = \mathbb{E}[|\langle Y_T^N, f_z \rangle|^2] \leq C b^{-6} \quad (4.15)$$

for $0 < b \leq \frac{1}{2}$.

Before we can do that, however, we need a long preliminary discussion. Indeed, Israelsson's proof of this fact in his Proposition 1 does not carry through immediately to the case of a general V , because it relies in an essential way on the bounds on characteristics. As explained in the Introduction though, the deterministic characteristics due to the $(\frac{1}{x})$ -potential, see §3.1, to which we can safely add the other transport generators without much change, yield the most singular contribution, so our strategy is to treat the non-local term $\mathcal{H}_{nonlocal} := \mathcal{H}_{V=0}^{ext} + \mathcal{H}_{V=0}^{(3)}$ as a *perturbation* of $\mathcal{H}_{transport}$, by using a Green function expansion. Note however the *twist* here: the operators $\mathcal{H}_{transport}^\kappa$ and $\mathcal{H}_{nonlocal}^\kappa$ are endomorphisms of $L^1(\Pi_{b_{max}})$, but $\mathcal{H}_{nonlocal}^{\kappa+1;\kappa}$ intertwines in some sense *two different copies* of $L^1(\Pi_{b_{max}})$, with different mappings, \mathcal{C}^κ , vs. $\mathcal{C}^{\kappa+1}$ to $L^1(\mathbb{R})$. The intertwining is not trivial, in the sense that $b\mathcal{H}_{nonlocal}^{\kappa+1;\kappa} \neq \mathcal{H}_{nonlocal}^\kappa$. This leads us to introduce the following operator-valued matrices.

Definition 4.3. 1. Let $L^1[\varepsilon](\Pi_{b_{max}}) := (\mathbb{R}[\varepsilon]/\varepsilon^3) \otimes L^1(\Pi_{b_{max}}) \simeq \varepsilon^0 \otimes L^1(\Pi_{b_{max}}) \oplus \varepsilon^1 \otimes L^1(\Pi_{b_{max}}) \oplus \varepsilon^2 \otimes L^1(\Pi_{b_{max}})$.

2. Let $\mathcal{H}[\varepsilon] : L^1[\varepsilon](\Pi_{b_{max}}) \rightarrow L^1[\varepsilon](\Pi_{b_{max}})$ be represented by the operator-valued matrix

$$\mathcal{H}[\varepsilon] := \begin{pmatrix} \mathcal{H}_{transport}^0 & 0 & 0 \\ \mathcal{H}_{nonlocal}^{1,0} & \mathcal{H}_{transport}^1 & 0 \\ 0 & \mathcal{H}_{nonlocal}^{2,1} & \mathcal{H}^2 \end{pmatrix}.$$

3. Let $\text{Ev} : L^1[\varepsilon](\Pi_{b_{max}}) \rightarrow L^1(\Pi_{b_{max}})$ be the evaluation mapping,

$$\text{Ev}(\varepsilon^0 \otimes h^0 + \varepsilon^1 \otimes h^1 + \varepsilon^2 \otimes h^2)(a, b) = h^0(a, b) + b h^1(a, b) + b^2 h^2(a, b). \quad (4.16)$$

The $(2, 2)$ -coefficient \mathcal{H}^2 – the sum $\mathcal{H}_{transport}^2 + \mathcal{H}_{nonlocal}^2$ – is coherent with the truncation. Another possibility (also coherent with Lemma 4.4 below, but introducing pointless complications) would be to consider the un-truncated infinite-dimensional matrix $\tilde{\mathcal{H}}[\varepsilon] :=$

$$\begin{pmatrix} \mathcal{H}_{transport}^0 & 0 & 0 & 0 & \cdots \\ \mathcal{H}_{nonlocal}^{1,0} & \mathcal{H}_{transport}^1 & 0 & 0 & \cdots \\ 0 & \mathcal{H}_{nonlocal}^{2,1} & \mathcal{H}^2 & 0 & \cdots \\ 0 & 0 & \mathcal{H}_{nonlocal}^{3,2} & \mathcal{H}_{transport}^3 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix} \text{ acting on } \tilde{L}^1[\varepsilon](\Pi_{b_{max}}) \equiv \mathbb{R}[\varepsilon] \otimes L^1(\Pi_{b_{max}}),$$

with evaluation mapping $\text{Ev}(\sum_{j \geq 0} \varepsilon^j \otimes h^j)(a, b) = \sum_{j \geq 0} b^j h^j$.

Lemma 4.4. Let $(h_t)_{0 \leq t \leq T} \in L^1[\varepsilon](\Pi_{b_{max}})$ be the solution of the time-evolution problem $\frac{dh_t}{dt} = \mathcal{H}[\varepsilon](t)h_t$ with terminal condition $h_T \equiv \varepsilon^0 \otimes h_T$.

Then $f_t := \mathcal{C}^0 \circ \text{Ev}(h_t)$ solves (1.33) with initial condition $\mathcal{C}^0(h_T)$.

Proof. By definition. □

Thus our time-evolution operator is $\mathcal{H}[\varepsilon]$. Let $\mathcal{H}_{transport}[\varepsilon] := \begin{pmatrix} \mathcal{H}_{transport}^0 & 0 & 0 \\ 0 & \mathcal{H}_{transport}^1 & 0 \\ 0 & 0 & \mathcal{H}_{transport}^2 \end{pmatrix}$

and $\mathcal{H}_{nonlocal}[\varepsilon] := \begin{pmatrix} 0 & 0 & 0 \\ \mathcal{H}_{nonlocal}^{1,0} & 0 & 0 \\ 0 & \mathcal{H}_{nonlocal}^{2,1} & \mathcal{H}_{nonlocal}^2 \end{pmatrix}$. The Green function first-order expansion then reads as follows,

$$U[\varepsilon](t, T) = U_{transport}[\varepsilon](t, T) - \int_t^T ds U[\varepsilon](t, s) \mathcal{H}_{nonlocal}[\varepsilon](s) U_{transport}[\varepsilon](s, T) \quad (4.17)$$

$U[\varepsilon](t, T)$, resp. $U_{transport}[\varepsilon](t, T)$ being the Green kernels (or evolution operators) obtained by integrating the time-inhomogeneous evolution systems generated by $\mathcal{H}[\varepsilon]$, resp. $\mathcal{H}_{transport}[\varepsilon]$, i.e. $u(t) = U[\varepsilon](t, T)u(T)$, resp. $u_{transport}(t) = U_{transport}[\varepsilon](t, T)u_{transport}(T)$, solves the linear equation $\frac{du(t)}{dt} = \mathcal{H}[\varepsilon](t)u(t)$, resp. $\frac{du_{transport}}{dt} = \mathcal{H}_{transport}[\varepsilon](t)u_{transport}(t)$. We shall actually require a *second-order* expansion of the Green kernel, obtained by iterating (4.17),

$$\begin{aligned} U[\varepsilon](t, T) &= U_{transport}[\varepsilon](t, T) - \int_t^T ds U_{transport}[\varepsilon](t, s) \mathcal{H}_{nonlocal}[\varepsilon](s) U_{transport}[\varepsilon](s, T) \\ &+ \int_t^T ds \int_t^s ds' U[\varepsilon](t, s') \mathcal{H}_{nonlocal}[\varepsilon](s') U_{transport}[\varepsilon](s', s) \mathcal{H}_{nonlocal}[\varepsilon](s) U_{transport}[\varepsilon](s, T) \end{aligned} \quad (4.18)$$

Thus (considering a terminal condition $h_T \equiv \varepsilon^0 \otimes h_T$)

$$\begin{aligned}
(\text{Ev} \circ U[\varepsilon](t, T))(h_T)(a, b) &= U_{transport}^0(t, T)h_T(a, b) \\
&- b \int_t^T ds U_{transport}^1(t, s)\mathcal{H}_{nonlocal}^{1,0}(s)U_{transport}^0(s, T)h_T(a, b) \\
&+ b^2 \int_t^T ds \int_t^s ds' U^2(t, s')\mathcal{H}_{nonlocal}^{2,1}(s')U_{transport}^1(s', s)\mathcal{H}_{nonlocal}^{1,0}(s)U_{transport}^0(s, T)h_T(a, b).
\end{aligned} \tag{4.19}$$

Fix $R > 0$ such that, for N large enough, the supports of the measures X_t^N , and the supports of the measure X_t , are $\subset [-R, R]$ for all $0 \leq t \leq 1$. Also, recall $b_{max} = \frac{1}{2}$. Define

$$\|f\|_{(L^1 \cap L^\infty)(\Pi_{b_{max}})} := \|f\|_{L^1(\Pi_{b_{max}})} + b_{max}R\|f\|_{L^\infty(\Pi_{b_{max}})} \tag{4.20}$$

and, for an operator $\mathcal{H} : (L^1 \cap L^\infty)(\Pi_{b_{max}}) \rightarrow (L^1 \cap L^\infty)(\Pi_{b_{max}})$,

$$\|\mathcal{H}\|_{(L^1 \cap L^\infty)(\Pi_{b_{max}})} := \sup_{\|f\|_{(L^1 \cap L^\infty)(\Pi_{b_{max}})}=1} \|\mathcal{H}f\|_{(L^1 \cap L^\infty)(\Pi_{b_{max}})}. \tag{4.21}$$

From the estimates proved in section 3, that is, from Lemmas 3.6, 3.7, 3.8, 3.9, we know that, for all $\kappa \in \mathbb{N}$ and $0 \leq t \leq s$:

$$\|U_{transport}^\kappa(t, s)\|_{L^1(\Pi_{b_{max}}) \rightarrow L^1(\Pi_{b_{max}})}, \|U_{transport}^\kappa(t, s)\|_{L^\infty(\Pi_{b_{max}}) \rightarrow L^\infty(\Pi_{b_{max}})} \leq 1; \tag{4.22}$$

$$\|\mathcal{H}_{nonlocal}^{\kappa+1;\kappa}\|_{(L^1 \cap L^\infty)(\Pi_{b_{max}})} = O(\|V'\|_{8+\kappa, [-3R, 3R]}R^{-1}), \tag{4.23}$$

$$\|\mathcal{H}_{nonlocal}^\kappa\|_{(L^1 \cap L^\infty)(\Pi_{b_{max}})} = O(\|V'\|_{7+\kappa, [-3R, 3R]}R^{-1}) \tag{4.24}$$

Hence

Lemma 4.5. *Let $T > 0$ fixed, and $0 \leq t \leq s \leq T$. Then $\|U^\kappa(t, s)\|_{(L^1 \cap L^\infty)(\Pi_{b_{max}})} \leq e^{C\|V'\|_{7+\kappa, [-3R, 3R]}R^{-1}t}$ for some constant $C > 0$.*

Proof. Results from (4.24), Tanabe [27], Theorem 4.4.1 (construction of fundamental solutions of temporally inhomogeneous equations) and Proposition 4.3.3 (bounded perturbations of generators of "stable" strongly continuous semi-groups, here of $\mathcal{H}_{transport}^2$ by $\mathcal{H}_{nonlocal}^2$). \square

Proof of (4.15). We fix $b_{max} = \frac{1}{2}$ and start from Itô's formula (1.8),

$$d\langle Y_t^N, f_t \rangle = \frac{1}{2}\left(1 - \frac{\beta}{2}\right)\langle X_t^N, f_t'' \rangle dt + \frac{1}{\sqrt{N}} \sum_{i=1}^N f_t'(\lambda_t^i) dW_t^i \tag{4.25}$$

with $f_T(x) = \frac{1}{x-z_T}$ represented as $(\mathcal{C}^0 h_T)(x)$, with h_T defined as in (2.28). Recall that $\|h_T\|_{L^1(\Pi_{b_{max}})} \approx 1/b_T$, $\|h_T\|_{L^\infty(\Pi_{b_{max}})} \approx 1/b_T^3$. One would like to say that, for $0 \leq t \leq T$, $\|h_t\|_{L^1(\Pi_{b_{max}})}, \|h_t\|_{L^\infty(\Pi_{b_{max}})}$ still have the same scaling in terms of powers of b_T ; however,

the boundary terms are bounded in terms of the L^∞ -norm of the restriction of h_t to the boundary, which implies a mixing of the L^1 and L^∞ -norms motivating the introduction of $\|\cdot\|_{(L^1 \cap L^\infty)(\Pi_{b_{max}})}$. Thus both $\|h_t\|_{L^1(\Pi_{b_{max}})}$ and $\|h_t\|_{L^\infty(\Pi_{b_{max}})}$ are bounded in terms of the worst term, i.e. the L^∞ -norm of h_T , which is $O(1/b_T^3)$. Again, this does not happen in Israelsson's case, or more generally (as can be proved) when V' grows linearly at infinity (see observation (i) in §3.7).

Integrating, we must bound three terms:

- (1) (initial condition) $\mathbb{E}|\langle Y_0^N, f_0 \rangle|^2$;
- (2) (drift term) $\mathbb{E} \left(\int_0^T dt |\langle X_t^N, f_t'' \rangle| \right)^2$;
- (3) (martingale term) $\mathbb{E} \left(\frac{1}{\sqrt{N}} \int_0^T dt \sum_{i=1}^N f_t'(\lambda_t^i) dW_t^i \right)^2$.

where $f_t = \mathcal{C}^0 h_t$. The time-evolution of h has been defined above; it depends on the parameter R , which must be chosen such that, for every $0 \leq t \leq T$, the supports of the measures X_t and X_t^N (for $N \geq N_0$ large enough) are $\subset [-R, R]$, which is equivalent to stating that $\Lambda^N := \sup_{0 \leq t \leq T} \max_{i=1, \dots, N} |\lambda_t^{N,i}| \leq R$, an event of probability $1 - O(e^{-c \frac{N}{\log_2(N)} V(R)})$ by Theorem 5.1.

Bounding (1) is easy but a bit awkward because one does not know the joint law of Λ^N and $M_0^N(z) - M_0(z)$, $z \in \Pi$. We use the 0-th order Stieltjes decomposition, $f_0(x) = (\mathcal{C}^0 h_0)(x) = \int da \int_{-b_{max}}^{b_{max}} (-ib) db f_z(x) h_0(a, b)$, together with the Cauchy-Schwarz inequality,

$$\begin{aligned} |\langle Y_0^N, f_0 \rangle|^4 &\leq \left| \int da \int_{-b_{max}}^{b_{max}} db |N| (M_0^N - M_0)(z) |h_0(a, b)| \right|^4 \\ &\leq \|h_0\|_{L^2(\Pi_{b_{max}})}^4 \|N|b|(M_0^N - M_0)(z)\|_{L^2(\Pi_{b_{max}})}^4. \end{aligned} \quad (4.26)$$

We use the obvious $L^1 - L^\infty$ -bound, $\|h_0\|_{L^2(\Pi_{b_{max}})}^4 \leq \|h_0\|_{L^1(\Pi_{b_{max}})}^2 \|h_0\|_{L^\infty(\Pi_{b_{max}})}^2$, insert the sum of characteristic functions, $1 = \mathbf{1}_{\Lambda^N \leq \Lambda} + \sum_{\ell \geq 1} \mathbf{1}_{2^{\ell-1}\Lambda \leq \Lambda^N \leq 2^\ell \Lambda}$ where $\Lambda := 3x_V \sqrt{2}$, and take $R = 2^\ell \Lambda$, $\ell = 0, 1, \dots$ on the event $\{2^{\ell-1}\Lambda \leq \Lambda^N \leq 2^\ell \Lambda\}$, yielding

$$\begin{aligned} \|h_0\|_{L^1(\Pi_{b_{max}})}^2 \|h_0\|_{L^\infty(\Pi_{b_{max}})}^2 &\leq (b_{max} R)^{-2} e^{4c\|V'\|_{7,[-3 \cdot 2^\ell \Lambda, 3 \cdot 2^\ell \Lambda]} (2^\ell \Lambda)^{-1} T} \|h_T\|_{(L^1 \cap L^2)(\Pi_{b_{max}})}^4 \\ &= (b_{max} R)^2 e^{4c\|V'\|_{7,[-3 \cdot 2^\ell \Lambda, 3 \cdot 2^\ell \Lambda]} (2^\ell \Lambda)^{-1} T} O(1/b_T^{12}) \end{aligned} \quad (4.27)$$

by (4.24). Then (again by the Cauchy-Schwarz inequality) we bound $\mathbb{E}|\langle Y_0^N, f_0 \rangle|^2$ by the sum

$$\mathbb{E}[\mathbf{1}_{\Lambda^N \leq \Lambda} |\langle Y_0^N, f_0 \rangle|^2] + \sum_{\ell=1}^{+\infty} \left(\mathbb{P}[2^{\ell-1}\Lambda \leq \Lambda^N \leq 2^\ell \Lambda] \right)^{1/2} \left(\mathbb{E}[\mathbf{1}_{2^{\ell-1}\Lambda \leq \Lambda^N \leq 2^\ell \Lambda} |\langle Y_0^N, f_0 \rangle|^4] \right)^{1/2}. \quad (4.28)$$

The factors $(b_{max} R)^2$ and $e^{4c\|V'\|_{7,[-3 \cdot 2^\ell \Lambda, 3 \cdot 2^\ell \Lambda]} (2^\ell \Lambda)^{-1} T}$ may be absorbed into the faster decreasing factor $O(e^{-\frac{1}{2}c' \frac{N}{\log_2(N)} V(2^{\ell-1}\Lambda)})$ coming from Assumption (vi) on V (see §1.3). There

remains to bound $\left(\mathbb{E} \left[\mathbf{1}_{2^{\ell-1}\Lambda \leq \Lambda^N \leq 2^\ell \Lambda} \|N|b|(M_0^N - M_0)(z)|\|_{L^2(\Pi_{b_{max}})}^4 \right] \right)^{1/2}$, and similarly $\mathbb{E} \left[\mathbf{1}_{\Lambda^N \leq \Lambda} \|N|b|(M_0^N - M_0)(z)|\|_{L^2(\Pi_{b_{max}})}^2 \right]$. This last remaining bound rests essentially on our rate of convergence Assumption (iii) in §1.3, but we need some decrease at real infinity away from the support to show that the integral in z converges. Hence, we bound the integral $\left(\int_{\Pi_{b_{max}}} (\cdots)^2 \right)^2$ by $O(R) \int_{[-2R, 2R] \times [-b_{max}, b_{max}]} (\cdots)^4 + O(1) \left(\int_{(\mathbb{R} \setminus [-2R, 2R]) \times [-b_{max}, b_{max}]} (\cdots)^2 \right)^2$. The integral over $[-2R, 2R] \times [-b_{max}, b_{max}]$ is $\leq O(R) \left(\int_{-2R}^{2R} da \int_{-b_{max}}^{b_{max}} db O(1) \right) = O(R^2)$; note that powers of R may be absorbed into the fast decreasing factor $O(e^{-\frac{1}{2}c' \frac{N}{\log_2(N)} V(2^{\ell-1}\Lambda)})$. As for the integral over $z \in (\mathbb{R} \setminus [-2R, 2R]) \times [-b_{max}, b_{max}]$, we first remark, using the standard order 0 Stieltjes decomposition with cut-off $b'_{max} = R$ (an unusual but convenient choice of cut-off in this particular context), that, on the event $\mathbf{1}_{2^{\ell-1}\Lambda \leq \Lambda^N \leq 2^\ell \Lambda}$,

$$N|M_0^N(z) - M_0(z)| = \langle Y_0^N, \chi_R f_z \rangle \quad (4.29)$$

and

$$\chi_R(x) f_z(x) = \int da' \int_{-R}^R (-ib') db' \frac{1}{x - z'} \mathcal{K}_{b_{max}}^0(\chi_R f_z)(a') \quad (4.30)$$

with (see (2.12)) $\mathcal{K}_{b_{max}}^0(\chi_R f_z)(a') = O(\frac{R^{-2}}{|z|}) O(\frac{1}{(1+(a'/R)^2)})$. Hence

$$\begin{aligned} & \mathbb{E} \|N|M_0^N(z) - M_0(z)|\|_{L^2((\mathbb{R} \setminus [-2R, 2R]) \times [-b_{max}, b_{max}])}^4 \\ &= O(R^{-6}) \mathbb{E} \left(\int_{(\mathbb{R} \setminus [-2R, 2R]) \times [-b_{max}, b_{max}]} \frac{da db}{|z|^2} \left| \int da' \int_{-R}^R db' \frac{(N|b'| |M_0^N(z') - M_0(z')|)^2}{1 + (|a'|/R)^2} \right|^2 \right)^2 \\ &= O(R^{-4}) \int \frac{da'}{1 + (|a'|/R)^2} \int_{-R}^R db' \mathbb{E} [(N|b'| |M_0^N(z') - M_0^N(z)|)^4] = O(R^{-2}). \end{aligned} \quad (4.31)$$

All together we have proved: $\mathbb{E} |\langle Y_0^N, f_0 \rangle|^2 = O(b_T^{-6})$.

Bounds for (2) and (3) are essentially pathwise, but more subtle and rely on our perturbative expansion for the Green kernel. As in (1), we bound $\mathbb{E}[\cdot]$ by a series in the index $\ell \in \mathbb{N}$ $\mathbb{E}[\mathbf{1}_{\Lambda^N \leq \Lambda} \cdot] + \sum_{\ell \geq 1} \mathbb{E}[\mathbf{1}_{\Lambda^N \geq 2^{\ell-1}\Lambda} \cdot]$ and take for R resp. $\Lambda, 2\Lambda, \dots, 2^\ell \Lambda, \dots$. Main terms are obtained by replacing f_t'', f_t' in (2), (3) with $(\mathcal{C}^0 u_t)''$, $(\mathcal{C}^0 u_t)'$, where $u_t := U_{transport}(t, T) h_T$. By assumption $h_T \geq 0$, so $\tilde{u}_t := \tilde{U}_{transport}(t, T) h_T = |u_t| \geq 0$ for $0 \leq t \leq T$, and these terms may be bounded as in Israelsson in a probabilistic way, by using the characteristic estimates proved in section 3.

Let us start by proving (2). First (using $|(f_z)''(x)| \leq \frac{2}{|x-z|^3} \leq \frac{2}{|b|} \frac{1}{|x-z|^2}$), we have pathwise

for fixed R

$$\begin{aligned}
\int_0^T dt |\langle X_t^N, (\mathcal{C}^0 u_t)'' \rangle| &\leq \int_0^T dt \left| \langle X_t^N, \int da \int_{-b_{max}}^{b_{max}} db |b| |(f_z)''(x)| u_t(a, b) \rangle \right| \\
&\leq 2 \int_0^T dt \int_{-3R}^{3R} da \int_{-b_{max}}^{b_{max}} db \frac{1}{b} \operatorname{Im} M_t^N(z) \tilde{u}_t(a, b) \\
&\leq 2 \int_0^T dt \int_{-3R}^{3R} da \int_{-b_{max}}^{b_{max}} db \frac{1}{b} \operatorname{Im} (M_t^N + M_t)(z) \tilde{u}_t(a, b) \\
&= 2 \int_0^T dt \int_{-3R}^{3R} da \int_{-b_{max}}^{b_{max}} db \partial_b [\operatorname{Im} (M_t^N + M_t)(z) \tilde{u}_t(a, b)] \ln(1/|b|) + \text{bdry}_1
\end{aligned} \tag{4.32}$$

where

$$\text{bdry}_1 := -2 \int_0^T dt \int_{-3R}^{3R} da [\operatorname{Im} (M_t^N + M_t)(a + ib_{max}) \tilde{u}_t(a, b_{max}) - (b_{max} \leftrightarrow -b_{max})] \ln(1/|b_{max}|) \tag{4.33}$$

is a boundary term.

Now we compare this expression to

$$\frac{8}{\beta} \int_{-3R}^{3R} da \int_{-b_{max}}^{b_{max}} db (\tilde{u}_T(a, b) - \tilde{u}_0(a, b)) \ln(1/|b|) \equiv \frac{8}{\beta} \int_0^T dt \frac{d}{dt} \left(\int_{-3R}^{3R} da \int_{-b_{max}}^{b_{max}} db \tilde{u}_t(a, b) \ln(1/|b|) \right). \tag{4.34}$$

The main terms in $\frac{8}{\beta} \frac{d}{dt} \tilde{u}_t(a, b)$ are those due to the $(\frac{1}{x})$ -kernel,

$$2 [\partial_a (\operatorname{Re} (M_t^N + M_t)(z) \tilde{u}_t(a, b)) + \partial_b (\operatorname{Im} (M_t^N + M_t)(z) \tilde{u}_t(a, b))].$$

The horizontal drift term $\partial_a \operatorname{Re} (M_t^N + M_t)$ vanishes by integration by parts up to a boundary term,

$$\text{bdry}_2 := 2 \int_0^T dt \int_{-b_{max}}^{b_{max}} db \ln(1/|b|) [\operatorname{Re} (M_t^N + M_t)(3R + ib) \tilde{u}_t(3R + ib) - (R \leftrightarrow -R)] \tag{4.35}$$

(note that $\operatorname{Re} (M_t^N + M_t)(\pm 3R + ib) = O(1/R)$ is bounded), while the vertical drift term is identical to (4.32). The other terms in $\mathcal{H}_{transport}(t)$, see (3.66), (3.67),

$$\partial_a \left((V'(a) - \frac{1}{2} V'''(a) b^2) \tilde{u}_t(a, b) \right) + (\partial_b (V''(a) b \tilde{u}_t(a, b))) + \text{bdry} \tag{4.36}$$

contribute respectively: yet another boundary term,

$$\text{bdry}_3 := \frac{8}{\beta} \int_0^T dt \int_{-b_{max}}^{b_{max}} db \ln(1/|b|) \left[(V'(3R) - \frac{1}{2} V'''(3R) b^2) \tilde{u}_t(3R, b) - (R \leftrightarrow -R) \right]; \tag{4.37}$$

and

$$\begin{aligned}
-\frac{8}{\beta} \int_0^T dt \int_{-3R}^{3R} da V''(a) \int_{-b_{max}}^{b_{max}} db \tilde{u}_t(a, b) \\
= O \left(\sup_{[-3R, 3R]} |V''| \right) e^{c \|V'\|_{7, [-3R, 3R]} R^{-1} T} \|u_T\|_{(L^1 \cap L^\infty)(\Pi_{b_{max}})} \tag{4.38}
\end{aligned}$$

plus a boundary term,

$$\text{bdry}_4 := \frac{8}{\beta} \int_0^T dt \int_{-3R}^{3R} da V''(a) [b_{max} \ln(1/|b_{max}|) \tilde{u}_t(a, b_{max}) - (b_{max} \leftrightarrow -b_{max})], \quad (4.39)$$

and also the integral over the domain $[-3R, 3R] \times [-b_{max}, b_{max}]$ of $(\frac{8}{\beta} \ln(1/|b|))$ times the boundary terms of §3.9. Finally, the contribution of the \tilde{c} -characteristic is known to be of the form $\frac{8}{\beta} \int_0^T dt \int_{3R}^{3R} da \int_{-b_{max}}^{b_{max}} db \tau_t(a, b) \tilde{u}_t(a, b) \ln(1/|b|) \geq 0$ with $\tau_t(\cdot, \cdot) \geq 0$, hence positive.

Using the $(L^1 \cap L^\infty)$ -bound of \tilde{u}_t , one sees that all boundary terms are $O\left(e^{c\|V'\|_{7,[-3R,3R]} R^{-1}T}\right) \|u_T\|_{(L^1 \cap L^\infty)(\Pi_{b_{max}})}$, times some rational expression of the type $b_{max}^\alpha (\ln(1/b_{max}))^\beta R^\gamma$ ($\alpha, \beta, \gamma \in \mathbb{Z}$), times some derivative of V , $\|V^{(j)}\|_{0,[-3R,3R]}$, $j = 1, 2, 3$, times possibly $\int_{-b_{max}}^{b_{max}} db \ln(1/|b|) = O(1)$. All these terms may be absorbed into the exponentially decreasing factor $\mathbb{P}[2^{\ell-1}\Lambda \leq \Lambda^N \leq 2^\ell\Lambda]$.

Consider now the *left-hand side* of (4.34). Considering the *adjoint evolution*, we get a time-reversed sub-Markov process (\bar{A}_t, \bar{B}_t) with kernel $p(s, \bar{a}_s; t, \bar{z}_t)$, $s \leq t$ (see §3.7). Since $t \mapsto |\bar{B}_t|$ decreases, we obtain

$$\begin{aligned} \left| \int da \int_{-b_{max}}^{b_{max}} db \tilde{u}_0(a, b) \ln(1/|b|) \right| &= \int da_T \int db_T u_T(a_T, b_T) \\ &\quad \int da_0 \int db_0 p(0, \bar{a}_0, \bar{b}_0; T, a_T, b_T) \ln(1/|\bar{b}_0|) \\ &\leq \ln(1/|b_T|) \int da_T \int db_T u_T(a, b) = O(\ln(1/|b_T|) b_T^{-1}) \end{aligned} \quad (4.40)$$

by the log-estimate (2.32). So much for the contribution of $U_{transport}^0$ to (2), which we have shown to be $O(1/b_T^6)$.

We now use the second-order expansion of the Green kernel (4.19). The *second term* in the expansion,

$$v(a, b) := \int_t^T ds U_{transport}^1(t, s) \mathcal{H}_{nonlocal}^{1,0}(s) U_{transport}^0(s, T) h_T(a, b) \quad (4.41)$$

leads to a development similar to (4.32):

$$\begin{aligned} &\int_0^T dt |\langle X_t^N, (\mathcal{C}^0 v_t)'' \rangle| \\ &\leq \int_0^T dt \left| \langle X_t^N, \int_{-3R}^{3R} da \int_{-b_{max}}^{b_{max}} db b^2 |(f_z)''(x)| \int_t^T ds (U_{transport}^1(t, s) |\mathcal{H}_{nonlocal}^{1,0}(s) u_s|)(a, b) \rangle \right| \\ &\leq 2 \int_0^T ds \left[\int_0^s dt \int_{-3R}^{3R} da \int_{-b_{max}}^{b_{max}} db |\text{Im} (M_t^N + M_t)(z)| \tilde{u}_t^s(a, b) \right] \\ &= -2 \int_0^T ds \left[\int_0^s dt \int_{-3R}^{3R} da \int_{-b_{max}}^{b_{max}} db \partial_b [\text{Im} (M_t^N + M_t)(z) \tilde{u}_t^s(a, b)] b \right] + \text{bdry}'_1 \end{aligned} \quad (4.42)$$

where $\tilde{u}_t^s(a, b) := \tilde{U}_{transport}^1(t, s)(|\mathcal{H}_{nonlocal}^{1,0}(s)u_s|)(a, b)$ (≥ 0), which we compare to

$$\begin{aligned} & \int_0^T ds \left[\frac{8}{\beta} \int_{-3R}^{3R} da \int_{-b_{max}}^{b_{max}} db (\tilde{u}_s^s(a, b) - \tilde{u}_0^s(a, b))b \right] \\ & \equiv \int_0^T ds \left[\frac{8}{\beta} \int_0^s dt \frac{d}{dt} \left(\int_{-3R}^{3R} da \int_{-b_{max}}^{b_{max}} db \tilde{u}_t^s(a, b)b \right) \right]. \end{aligned} \quad (4.43)$$

The right-hand side in (4.43) decomposes in the same way as explained below (4.36) – but with $\kappa = 1$ now –. Compared to the first term in the expansion, $\ln(b)$ has been replaced with b (which may simply be bounded by a constant, $b \leq \frac{1}{2}$), so logarithms disappear in the estimates, while the replacement of u_s by $|\mathcal{H}_{nonlocal}^{1,0}(s)u_s|$ produces the supplementary factor $|||\mathcal{H}_{nonlocal}^{1,0}(s)|||_{(L^1 \cap L^\infty)(\Pi_{b_{max}})}$, which is controlled by $\mathbb{P}[2^{\ell-1}\Lambda \leq \Lambda^N \leq 2^\ell\Lambda]$. The total contribution to (2) is therefore simply $O(b_T^{-6})$.

The *last* term in the Green kernel expansion,

$$w(a, b) := \int_t^T ds \int_t^s ds' U^2(t, s') \mathcal{H}_{nonlocal}^{2,1}(s') U_{transport}^1(s', s) \mathcal{H}_{nonlocal}^{1,0}(s) U_{transport}^0(s, T) h_T(a, b), \quad (4.44)$$

leads now to a third contribution which is bounded in a very simple way,

$$\begin{aligned} & \int_0^T dt |\langle X_t^N, (\mathcal{C}^0 w_t)'' \rangle| \leq \int_0^T dt \left| \langle X_t^N, \int_{-3R}^{3R} da \int_{-b_{max}}^{b_{max}} db |b|^3 |(f_z)''(x)| \right. \\ & \quad \left. \int_t^T ds \int_t^s ds' \left| U^2(t, s') \mathcal{H}_{nonlocal}^{2,1}(s') (U_{transport}^1(s', s) \mathcal{H}_{nonlocal}^{1,0}(s) u_s)(a, b) \right| \right| \end{aligned} \quad (4.45)$$

Since $|b|^3 |(f_z)''(x)| = O(1)$, (4.45) is simply bounded in the end by $\|u_T\|_{(L^1 \cap L^\infty)(\Pi_{b_{max}})}$, times the product of the $(L^1 \cap L^\infty)(\Pi_{b_{max}})$ - operator norms $|||U_{transport}^i(\cdot, \cdot)|||$, $|||\mathcal{H}_{nonlocal}^{i+1,i}(\cdot)|||$ ($i = 0, 1$) figuring in the integral, which is once again controlled by $\mathbb{P}[2^{\ell-1}\Lambda \leq \Lambda^N \leq 2^\ell\Lambda]$, yielding a total contribution $O(b_T^{-6})$ to (2) as for the previous term.

As a simple consequence of these calculations, we also get a similar bound for the martingale term (3). First, by the Cauchy-Schwarz, and then Burkholder-Davis-Gundy inequalities (see e.g. [15], Theorem 3.28),

$$\begin{aligned} & \left| \mathbb{E} \left[\mathbf{1}_{2^{\ell-1}\Lambda \leq \Lambda^N \leq 2^\ell\Lambda} \left(\frac{1}{\sqrt{N}} \int_0^T dt \sum_{i=1}^N f'_t(\lambda_t^i) dW_t^i \right)^2 \right] \right| \\ & \leq C \left(\mathbb{P}[2^{\ell-1}\Lambda \leq \Lambda^N \leq C2^\ell\Lambda] \right)^{1/2} \left(\mathbb{E} \left[\left(\int_0^T dt \langle X_t^N, (f'_t)^2 \rangle dt \right)^2 \right] \right)^{1/2}. \end{aligned} \quad (4.46)$$

Now, using $|(f_z)'(x)| \leq \frac{1}{|b|} \frac{1}{|x-z|}$ and (7.12), we get

$$\begin{aligned}
& \int_0^T dt |\langle X_t^N, ((\mathcal{C}^0 h_t)')^2 \rangle| dt \\
& \leq \int_0^T dt \int da \int_0^{+\infty} db \cdot \int da' \int_{-b_{max}}^{b_{max}} db' |h_t(a, b)| |h_t(a', b')| |\langle X_t^N, |b f'_z(\cdot)| |b' f'_{z'}(\cdot)| \rangle| \\
& \leq \int_0^T dt \int da \int_0^{+\infty} db \cdot \int da' \int_{-b_{max}}^{b_{max}} db' |h_t(a, b)| |h_t(a', b')| |\langle X_t^N, b^2 |f'_z(\cdot)|^2 \rangle| \\
& \leq \sup_{0 \leq t \leq T} \left(\int da' \int_{-b_{max}}^{b_{max}} db' |h_t(a', b')| \right) \cdot \int_0^T dt \left(\int da \int_{-b_{max}}^{b_{max}} db \frac{1}{|b|} \text{Im } M_t^N(z) |h_t(a, b)| \right). \tag{4.47}
\end{aligned}$$

The second factor in (4.47) is bounded like the drift term (2), while the first one is just $\|h_t\|_{L^1(\Pi_{b_{max}})}$. All together, the martingale term is bounded by $O(b_T^{-6})$ like the previous ones. \square

5 Large deviation bounds for the support of the measure

As a key technical argument required for the convergence of our scheme, we prove in this section the following bound for the probability that the support of the measure is large. Since the number N of eigenvalues varies in this section, we emphasize the N -dependence of the process when we judge it necessary by writing $\lambda_t^{N,i}$ instead of λ_t^i . Recall the constant $x_V^0 > 0$ from Assumption (v) on V in §1.3. Below, see Lemma 5.4, we define another constant $R_V > 0$ if some entropic condition on the initial measure holds, implying the initial support large deviation estimate, Assumption (i) in §1.3 with the same constant R_V – which may also be taken as a more general condition on the initial measure.

Lemma 5.1. *Let V satisfy the Assumptions (i)-(vi) of §1.3. Then:*

1. *There exist $x_V \geq \max(R_V, x_V^0, \sqrt{\beta/2})$ and $c_V \in [0, 1)$ such that:*

- (i) $\forall x \in \mathbb{R} \mid |x| \geq x_V, |V'(x)| \geq \frac{\beta^2}{\sqrt{2}-1};$
- (ii) $\forall x \in \mathbb{R} \mid |x| \leq x_V, V(x) \leq V(x_V);$
- (iii) $\forall x \geq x_V, \left| \frac{V'(x)}{V'(-x)} + 1 \right| \leq c_V.$

2. $\forall x \in \mathbb{R}, xV'(x) \geq (1 - c_V)(V(|x|) - V(x_V)).$

Proof.

1. Elementary. In order to optimize subsequent bounds one may choose

$c_V \in \left(\limsup_{x \rightarrow +\infty} \left| \frac{V'(x)}{V'(-x)} + 1 \right|, 1 \right)$ and take $x_V \geq x_V^0$ minimal such that it satisfies the above conditions.

2. Immediate consequence of convexity of V and of conditions (ii), (iii) (condition (i) is required only in the estimates of §5.2); note in particular that $\text{sgn}(V'(x)) = \text{sgn}(x)$ for $|x| \geq x_V$. Hence: if $x \geq 0$ then $xV'(x) \geq (x - x_V)V'(x) \geq V(x) - V(x_V) \geq (1 - c_V)(V(x) - V(x_V))$. If $-x_V \leq x \leq 0$ then $xV'(x) \geq 0 \geq (1 - c_V)(V(|x|) - V(x_V))$. Finally, if $x \leq -x_V$, then $xV'(x) \geq (x - (-x_V))V'(x) \geq (1 - c_V)(|x| - x_V)V'(|x|) \geq (1 - c_V)(V(|x|) - V(x_V))$.

□

Theorem 5.1. *Let $p = 1, 2, \dots$ and $T > 0$. Assume the initial support large deviation estimate condition (i) of §1.3 holds. Then there exist some constant $c > 0$ and $N_0 \in \mathbb{N}^*$ such that, for $N \geq N_0$,*

$$\mathbb{P}[\sup_{0 \leq t \leq T} \max_{i=1, \dots, N} |\lambda_t^{N,i}| > R\sqrt{2}] \leq e^{-c \frac{N}{\log_2(N)} (V(R) - V(x_V))} \quad (5.1)$$

for any $R > 3x_V$.

This bound is based on an *inductive argument on the number of eigenvalues* based on the comparison principle for one-dimensional stochastic differential equations (see e.g. [15], chap. 5, Prop. 2.18) and on the Assumptions on V made in §1.3.

The reader may wonder whether the above initial support large deviation estimate hypothesis holds for a large class of initial conditions. In the next paragraph (§5.1) we show that it does hold for N -dependent initial conditions with uniformly bounded *Tsallis entropy* with respect to equilibrium measures $\mu_{eq, \tilde{V}}^N$, where \tilde{V} is not too "far" from V (actually, \tilde{V} may even depend on N).

5.1 The entropic argument

The first paragraph of this section is based on the book by C. Kipnis and C. Landim [17], Appendix 1, §9 for general considerations on entropies, and on the article by Johansson [14] for applications to random matrices. Recall from (1.3) the equilibrium measure,

$$d\mu_{eq,V}^N(\{\lambda^i\}_i) = \rho_V^N(\{\lambda^i\}_i) d\lambda^1 \cdots d\lambda^N, \quad \rho_V^N(\{\lambda^i\}) := \frac{1}{Z_V^N} \left(\prod_{j \neq i} |\lambda^j - \lambda^i| \right)^{\beta/2} \exp \left(-2N \sum_{i=1}^N V(\lambda^i) \right). \quad (5.2)$$

We added a lower index $\mu_{eq,V}^N$ to μ_{eq}^N because (as we shall see) we shall be able to consider initial conditions μ_0^N with uniformly bounded entropy with respect to possibly various equilibrium measures μ_{eq,V^N}^N , where V^N is "close enough" to V . The advantage is that the deterministic limit ρ_0 is not necessarily the equilibrium density ρ_{eq} .

Definition 5.2 (entropies). *For $p = 1, 2, \dots$, we let, for a measure $\mu = \mu(\{\lambda^i\}_{i=1, \dots, N})$ with density $\frac{d\mu}{d\mu_{eq}^N} = \frac{\rho}{\rho_V^N}$ with respect to some reference measure μ_{ref} ,*

$$H_{2p}(\mu|\mu_{ref}) := \int d\mu_{ref}(\{\lambda^i\}_i) \left(\frac{d\mu}{d\mu_{ref}}(\{\lambda^i\}_i) \right)^{2p/(2p-1)}. \quad (5.3)$$

This family of entropies is a particular case (commonly called *Tsallis entropies*) of a more general definition depending on a function $\phi : \mathbb{R} \rightarrow \mathbb{R}$,

$$H_\phi(\mu|\mu_{ref}) := \int d\mu_{ref}(\{\lambda^i\}_i) \phi\left(\frac{d\mu}{d\mu_{ref}}(\{\lambda^i\}_i)\right). \quad (5.4)$$

Since $\frac{x^{q+1}-x}{q} \xrightarrow{q \rightarrow 1} x \ln(x)$, the usual entropy – for which $\phi(x) = x \ln(x)$ – may be considered as the limit of the above family when $p \rightarrow \infty$. The case when $\mu_{ref} = \mu_{eq}^N$ is the equilibrium measure with respect to the dynamics (1.1) is particularly interesting. Namely, let $\mu_t = \mu_t(\{\lambda^i\}_{i=1,\dots,N})$ be the measure of the process with initial measure μ . As shown in [17], Proposition 9.1, $t \mapsto H_\phi(\mu_t; \mu_{eq}^N)$ decreases provided ϕ is *convex*.

The following characterization of the entropy as the solution of a problem of optimization generalizes a well-known formula for the usual entropy found in [17], Appendix 1, §8, :

Lemma 5.3. *Let $p = 1, 2, \dots$ and $\mu = \mu(\{\lambda^i\}_i)$ be measure with density $\frac{d\mu}{d\mu_{ref}}$ with respect to μ_{ref} . Let*

$$\Phi_{2p}(\mu; f) := \langle \mu, f \rangle - (\langle \mu_{ref}, f^{2p} \rangle)^{1/2p} \quad (5.5)$$

for $f : \mathbb{R}^N \rightarrow \mathbb{R}$ function of λ^i , $i = 1, \dots, N$. Then

$$\sup_{f: \mathbb{R}^N \rightarrow \mathbb{R} | \langle \mu_{ref}, f^{2p} \rangle = 1} \Phi_{2p}(\mu; f) = (H_{2p}(\mu))^{(2p-1)/2p} - 1. \quad (5.6)$$

We include a proof since we did not find one in the literature.

Proof.

- (i) We first prove that the functional $\Phi(\cdot) := \Phi_{2p}(\mu; \cdot)$ is *concave*, which implies that local extrema are maxima. Namely, let $f, g : \mathbb{R}^N \rightarrow \mathbb{R}$ and $t \in [0, 1]$. We must prove that $\Phi((1-t)f + tg) \geq (1-t)\Phi(f) + t\Phi(g)$, or equivalently,

$$\begin{aligned} & \int d\mu_{ref}(\boldsymbol{\lambda}) ((1-t)f(\boldsymbol{\lambda}) + tg(\boldsymbol{\lambda}))^{2p} \\ & \leq \left((1-t) \left(\int d\mu_{ref}(\boldsymbol{\lambda}) (f(\boldsymbol{\lambda}))^{2p} \right)^{1/2p} + t \left(\int d\mu_{ref}(\boldsymbol{\lambda}) (g(\boldsymbol{\lambda}))^{2p} \right)^{1/2p} \right)^{2p} \end{aligned} \quad (5.7)$$

Expand both sides in (5.7) using the binomial formula and consider the coefficient of $(1-t)^k t^{2p-k}$, $k = 0, 1, \dots, 2p$. It turns out that the inequality is actually valid coefficient by coefficient, i.e. that

$$\int d\mu_{ref}(\boldsymbol{\lambda}) (f(\boldsymbol{\lambda}))^k (g(\boldsymbol{\lambda}))^{2p-k} \leq \left(\int d\mu_{ref}(\boldsymbol{\lambda}) (f(\boldsymbol{\lambda}))^{2p} \right)^{k/2p} \left(\int d\mu_{ref}(\boldsymbol{\lambda}) (g(\boldsymbol{\lambda}))^{2p} \right)^{(2p-k)/2p}, \quad (5.8)$$

which follows from Hölder's inequality.

- (ii) We find the unique local extremum by means of a Lagrange multiplier, namely, we consider local extrema for

$$\tilde{\Phi}(f, \varepsilon) := \Phi(f) - \varepsilon(\langle \mu_{ref}, f^{2p} \rangle - 1). \quad (5.9)$$

Differentiating in some arbitrary direction $h : \mathbb{R}^N \rightarrow \mathbb{R}$ and using the constraint, one obtains the condition

$$\langle \mu, h \rangle - \langle \mu_{ref}, f^{2p-1} h \rangle = 2p\varepsilon \langle \mu_{ref}, f^{2p-1} h \rangle \quad (5.10)$$

from which $\frac{d\mu}{d\mu_{ref}} = (2p\varepsilon + 1)f^{2p-1}$, thus defining uniquely the location f of the absolute maximum. The constraint $\langle \mu_{ref}, f^{2p} \rangle = 1$ yields $2p\varepsilon + 1 = \left(\langle \mu_{ref}, (\frac{d\mu}{d\mu_{ref}})^{2p/(2p-1)} \rangle \right)^{(2p-1)/2p}$, whence $\Phi(f) = \left(\langle \mu_{ref}, (\frac{d\mu}{d\mu_{ref}})^{2p/(2p-1)} \rangle \right)^{(2p-1)/2p} - 1$.

□

We may now state the main result of this paragraph. *Let $(V^N)_{N \geq 1}$ be a family of convex potentials such that $V^N(x) \rightarrow +\infty$ when $|x| \rightarrow \infty$, and there exist constants $R_V^0, C > 0$ such that*

$$\min_{|x| \geq R_V^0} V(x) > 0; \quad \forall x \in \mathbb{R} \mid |x| \geq R_V^0, \quad C^{-1} \leq \frac{V(x)}{V^N(x)} \leq C. \quad (5.11)$$

Lemma 5.4. *Assume the initial measures μ_0^N are such that $H := \sup_{N \geq 1} H_{2p}(\mu_0^N | \mu_{eq, V^N}^N) < \infty$. Then there exists $R_V \geq R_V^0$ and some constant $c > 0$, both depending only on V , such that $R \geq R_V$,*

$$\mathbb{P}[\max_{i=1, \dots, N} |\lambda_t^i| > R] \leq H^{(2p-1)/2p} e^{-cNV(R)}. \quad (5.12)$$

Letting $N \rightarrow \infty$ as using Assumption (ii) on the initial measure in §1.3, we deduce: $\text{supp}(\rho_0) \subset [-R_V, R_V]$.

Proof. We use Lemma 5.3. As proved in [14], Lemma 4.4 (take $M_N = N$ and $h \equiv 0$),

$$u_{eq}^N(x) \leq C^N e^{-\frac{1}{2}\beta N(V^N(x) - 2\log(1+x^2))}, \quad (5.13)$$

where $u_{eq, V^N}^N(x) := \int d\lambda^2 \cdots \int d\lambda^N \rho_{eq, V^N}^N(x, \lambda^2, \dots, \lambda^N)$ is the one-point equilibrium density. (Note that, by definition, $\mu_t^N = \mu_t^N(\{\lambda_t^i\})$ is assumed to be symmetric under any permutation of the eigenvalues.) Since $V(x) \rightarrow +\infty$ when $|x| \rightarrow \infty$ and V is convex, V grows at least linearly at infinity, hence (by our assumptions on $(V^N)_{N \geq 1}$) one has the simpler bound $u_{eq}^N(x) \leq C^N e^{-\frac{1}{4}\beta NV(x)}$, with a possibly different constant C depending on V . Taking $f(\{\lambda^i\}_i) := e^{cNV(R)} \mathbf{1}_{|\lambda^i| > R}$ ($c > 0$), we get

$$\langle \mu_{eq, V^N}^N, f^{2p} \rangle = e^{2pcNV(R)} \int_{|x| > R} u_{eq, V^N}^1(x) dx \leq 1 \quad (5.14)$$

for R large enough and c small enough. One may arrange to have the equality $\langle \mu_{eq, V^N}^N, f^{2p} \rangle = 1$ and obtains by Lemma 5.3

$$\langle \mu_0^N, f \rangle \leq (H_{2p}(\mu_0^N | \rho_{eq, V^N}^N))^{(2p-1)/2p} \leq H^{(2p-1)/2p}. \quad (5.15)$$

From this inequality follows

$$\mathbb{P}[\max_{i=1,\dots,N} |\lambda_0^i| > R] \leq \sum_i \mathbb{P}[|\lambda_0^i| > R] = N e^{-cNV(R)} \langle \mu_0^N, f \rangle \leq N H^{(2p-1)/2p} e^{-cNV(R)}. \quad (5.16)$$

By taking R large enough and c smaller, one may absorb the factor N in the exponential. \square

If $V^N \equiv V$ in the above Lemma, then (by the entropy decrease property) (5.12) also holds for $\max_{i=1,\dots,N} |\lambda_t^i|$, for all $t \geq 0$. We only need the above result for $t = 0$ in the sequel, but this uniform in t bound is pleasant in itself. Roughly speaking (as proved in the upcoming subsection), under further assumptions on V , we may insert a supremum $\sup_{0 \leq t \leq 1}$ inside the left-hand side of (5.12). (As argued in Lemma 5.5 below, $\sup_{0 \leq t < T} \max_{i=1,\dots,N} |\lambda_t^i|$ increases with T and is a.s. unbounded if $T = +\infty$). The proof relies, however, on wholly different arguments.

5.2 The inductive argument

We now proceed to prove Theorem 5.1. As part of the argument, we use the results of §5.1 to control the initial condition. Let us start with the following

Lemma 5.5. *Let $(R_t)_{t \geq 0}$ be the solution of the SDE*

$$dR_t = \frac{1}{\sqrt{N}} dB_t + \left(\frac{C}{R_t} - V'(R_t) \right) dt, \quad (5.17)$$

with initial condition $R_0 \equiv r_0 > 0$, where $(B_t)_{t \geq 0}$ is a standard Brownian motion, V is a function satisfying Assumptions (i)-(vi) of §1.3, and $0 \leq C \leq 2\beta$. Then, for $A > \max(R_0, \frac{3}{2}\sqrt{2\beta})$,

$$\mathbb{P}[\sup_{t \in [0,1]} R_t > A] \leq \frac{1}{(A - r_0)^2} \exp[-cN(V(A) - V(\max(r_0, x_V)))] . \quad (5.18)$$

Proof. The argument is based on the known distribution of first-passage times for Brownian motion with constant drift $-\mu > 0$; as shown in e.g. [15], p. 197, if $\mu, A > 0$ and B is a standard Brownian motion issued from 0, then

$$\mathbb{P}[\sup_{t > 0} (B_t - \mu t) > A] = e^{-2\mu A}. \quad (5.19)$$

The idea is in principle to apply this result to excursions above level κA , where $\kappa \in (\max(\frac{r_0}{A}, \frac{2}{3}), 1)$, during which one has by assumption $\frac{C}{r} - V'(r) \leq -\frac{1}{2}V'(r)$ and $V'(r) \geq V'(\kappa A)$, whence the drift in (5.17) is $\leq -\mu := -\frac{1}{2}V'(\kappa A)$. However, the drift is *not* constant in our case; rather, what happens roughly is that R_t oscillates in a vicinity of the equilibrium point defined as the unique solution r_{eq} of the equation $\frac{C}{r} = V'(r)$, with independent upper, resp. lower excursions above, resp. below r_{eq} . Because these excursions are independent, the process $(R_t)_{t \geq 0}$ is a.s. *unbounded*, so the above argument needs some refinements. Actually, this is typically a situation where the Freidlin-Wentzell theory (see [10], or [5], chap. 5, §7) applies, yielding explicit large deviation estimates for the exit time from the domain $[0, A]$ around its means, which is $\approx e^{NF_V(r_0, A)}$ for N large for some explicit functional F_V .

We do not believe that Freidlin-Wentzell theory applies directly to our situation since we restrict to a finite time window, so we shall rely on a homemade proof, which is however, not surprisingly, very reminiscent of the arguments found in [5].

The proof relies on inductive arguments based on (5.19) and first hitting times T_A , $T_{\kappa A}$ and $T_{(2\kappa-1)A}$ for *three* levels, $A > \kappa A > (2\kappa - 1)A$. We define more precisely $T_r := \inf\{t \in [0, 1] \mid R_t = r\}$ for the process $(R_t)_{t \geq 0}$ of (5.17) issued from r_0 , and similarly (after time-translation θ_τ of the paths by some random waiting time τ) $T_r^{r_0} \circ \theta_\tau := \inf\{t \in [0, 1 - \tau] \mid R_{t+\tau} = r\}$.

The first step is the following. If $T_A \leq 1$ then (given that $r_0 \leq \kappa A$ by construction) $T_{\kappa A} \leq 1$. Let

$$\tau_1 := \min \left(1, T_{\kappa A} + \frac{1 - \kappa}{2V'(\kappa A)} \right). \quad (5.20)$$

Then

$$\begin{aligned} \mathbb{P}^{r_0} \left[\sup_{t \in [0, 1]} R_t > A \right] &= \mathbb{P}[T_A \leq 1] \\ &= \mathbb{P}[(T_A \leq 1) \cap (T_{\kappa A} \leq 1) \cap \Omega] + \mathbb{P}[(T_A \leq 1) \cap (T_{\kappa A} \leq 1) \cap \Omega^c], \end{aligned} \quad (5.21)$$

where Ω is the event

$$\Omega := \left\{ (T_A \leq 1) \cap (T_{\kappa A} \leq 1) \cap \left(\sup_{T_{\kappa A} \leq t \leq \tau_1} |B_t - B_{T_{\kappa A}}| \leq \frac{1 - \kappa}{2} A \sqrt{N} \right) \right\}. \quad (5.22)$$

Under Ω we can prove by an elementary argument that $\min_{[T_{\kappa A}, \tau_1]} R \geq (2\kappa - 1)A$. Namely, assuming $\tau := \inf\{T_{\kappa A} \leq t \leq \tau_1 \mid R_t \leq (2\kappa - 1)A\} < \tau_1$, we get a contradiction because $R_\tau = \kappa A + (R_\tau - R_{T_{\kappa A}}) \geq \kappa A + \frac{1}{\sqrt{N}}(B_\tau - B_{T_{\kappa A}}) - (\tau - T_{\kappa A})V'(\kappa A) > (2\kappa - 1)A$. The complementary event Ω^c has small probability, typically of the form we want in the end,

$$\mathbb{P}[(T_A \leq 1) \cap (T_{\kappa A} \leq 1) \cap \Omega^c] \leq \exp \left(-c \left(\frac{1 - \kappa}{2} A \sqrt{N} \right)^2 / (\tau_1 - T_{\kappa A}) \right) \leq \exp \left(-c \frac{1 - \kappa}{2} N A V'(\kappa A) \right) \quad (5.23)$$

hence we leave it aside and come back to

$$\begin{aligned} \mathbb{P}[(T_A \leq 1) \cap (T_{\kappa A} \leq 1) \cap \Omega] &= \mathbb{P}[(T_A \leq 1) \cap (T_{\kappa A} \leq 1) \cap \Omega \cap \Omega'] \\ &\quad + \mathbb{P}[(T_A \leq 1) \cap (T_{\kappa A} \leq 1) \cap \Omega \cap (\Omega')^c], \end{aligned} \quad (5.24)$$

where

$$(\Omega')^c = \{T_A \leq \tau_1\}. \quad (5.25)$$

Under the event $(T_A \leq 1) \cap (T_{\kappa A} \leq 1) \cap \Omega \cap (\Omega')^c$, the process $R|_{[T_{\kappa A}, \tau_1]}$ is issued from κA , hits A and has drift $\leq -\frac{1}{2}V'((2\kappa - 1)A)$ since $(2\kappa - 1)A \geq \frac{4}{3} \geq x_V$ by assumption, whence (by the SDE comparison principle)

$$\begin{aligned} \mathbb{P}[(T_A \leq 1) \cap (T_{\kappa A} \leq 1) \cap \Omega \cap (\Omega')^c] &\leq \mathbb{P} \left[\sup_{t \geq 0} \left(\frac{B_t}{\sqrt{N}} - V'((2\kappa - 1)A)t \right) \geq (1 - \kappa)A \right] \\ &\leq \exp \left(-(1 - \kappa)N A V'((2\kappa - 1)A) \right). \end{aligned} \quad (5.26)$$

Again, we leave aside this case and are left with

$$\begin{aligned} \mathbb{P}[(T_A \leq 1) \cap (T_{\kappa A} \leq 1) \cap \Omega \cap \Omega'] &= \mathbb{P}[(T_A \leq 1) \cap (T_{\kappa A} \leq 1) \cap \Omega \cap \Omega' \cap (\Omega'')^c] \\ &+ \mathbb{P}[(T_A \leq 1) \cap (T_{\kappa A} \leq 1) \cap \Omega \cap (\Omega') \cap (T_{\kappa A} \circ \theta_{\tau_1} \leq 1 - \tau_1)], \end{aligned} \quad (5.27)$$

where

$$(\Omega'')^c := \left\{ \min_{[\tau_1, T_A]} R > \kappa A \right\}. \quad (5.28)$$

The event $(T_A \leq 1) \cap (T_{\kappa A} \leq 1) \cap \Omega \cap \Omega' \cap (\Omega'')^c$ has probability bounded as in (5.26).

We now iterate this procedure under the event $(T_A \leq 1) \cap (T_{\kappa A} \leq 1) \cap \Omega \cap (\Omega') \cap (T_{\kappa A} \circ \theta_{\tau_1} \leq 1 - \tau_1)$, starting from time $T_{\kappa A} \circ \theta_{\tau_1}$ instead of $T_{\kappa A}$, obtaining thus an increasing sequence of waiting times $\tau_1 < \tau_2 < \dots$, with by construction $\tau_{k+1} - \tau_k \geq \frac{(1-\kappa)A}{2V'(\kappa A)}$. The number of iterations is $\leq \frac{2V'(\kappa A)}{(1-\kappa)A}$, and at each iteration we leave aside the sum of three terms, each bounded by $\exp(-c(1-\kappa)NAV'((2\kappa-1)A))$, whence, for some constant $C > 0$,

$$\mathbb{P}[\sup_{0 \leq t \leq 1} R_t > A] \leq C \frac{V'(\kappa A)}{(1-\kappa)A} \exp(-c(1-\kappa)NAV'((2\kappa-1)A)). \quad (5.29)$$

Using the regular growth assumption on V , we deduce:

$$\mathbb{P}[\sup_{0 \leq t \leq 1} R_t > A] \leq C \frac{1}{((1-\kappa)A)^2} \exp(-c'(1-\kappa)NAV'(A)). \quad (5.30)$$

Fix $\kappa_0 = \max(\frac{R_0}{A}, \frac{2}{3})$. Using the convexity bound $(A - r_0)V'(A) \geq V(A) - V(r_0)$ if $r_0 \geq \frac{2}{3}A \geq x_V$, and $\frac{1}{3}AV'(A) \geq (A - x_V)V'(A) \geq V(A) - V(x_V) \geq V(A) - V(\max(r_0, x_V))$ otherwise, we obtain the desired result.

Note that these arguments also apply to the Brownian motion with constant drift reflected at 0, for which we found no explicit result in the literature. \square

From this one easily deduces a large deviation estimate $\mathbb{P}[\sup_{t \in [0,1]} \lambda_t^N > A]$ for the *smallest* eigenvalue λ_t^N , since the other eigenvalues drive λ_t^N to the left. Then, assuming $(\lambda_t^i)_{i=1, \dots, N}$ are in decreasing order, i.e. $\lambda^N < \lambda^{N-1} < \dots < \lambda^2 < \lambda^1$, we may try to derive inductively large deviation estimates for $\lambda^{N-1}, \dots, \lambda^1$, following the proof of Lemma 5.5. We tried this and found a bound in $O(\log N)$ at best for $\sup_{[0,1]} \lambda_t^1$. This was to be expected since – assuming spacings $\lambda^j - \lambda^{j+1} \approx 1/N$ are of the expected order – the rightward force exerted on λ^k by the eigenvalues $\lambda^N, \dots, \lambda^{k+1}$ to the left of λ^k is $O(\log(N-k))$. The reason why this bound is *not* correct is that the *total* force exerted on λ^k by *all* eigenvalues $(\lambda^j)_{j \neq k}$ is really $O(1)$ and not $O(\log N)$, because the leftward force exerted by the eigenvalues to the *right* of λ^k nearly equilibrates that exerted by the eigenvalues to the left. We are unable to prove this directly for each eigenvalue, though. In order to bypass this difficult issue, we use the explicit cancellations occurring for the *averaged* quantity

$$\mathcal{V}_t^0 := \frac{1}{2N} \sum_{i=1}^N (\lambda_t^i)^2 \quad (5.31)$$

(the upper index 0 means that this quantity appears at step #0 of an induction explained below). Following computations due to Rogers and Shi [26] (see also [18]), we find

$$d\mathcal{V}_t^0 = \frac{1}{N\sqrt{N}} \sum_{i=1}^N \lambda_t^i dW_t^i + \left(\frac{1}{2N} + \frac{\beta}{2N^2} \sum_{1 \leq i \neq j \leq N} \frac{\lambda_t^i}{\lambda_t^i - \lambda_t^j} \right) dt - \frac{1}{N} \sum_{i=1}^N V'(\lambda_t^i) \lambda_t^i dt. \quad (5.32)$$

Then:

- (i) (by P. Lévy's theorem) $\sum_{i=1}^N \lambda_t^i dW_t^i \stackrel{(law)}{=} \sqrt{2N\mathcal{V}_t^0} dB_t$, where $(B_t)_{t \geq 0}$ is a standard Brownian motion;
- (ii) (by antisymmetry) $\frac{\beta}{2N^2} \sum_{1 \leq i \neq j \leq N} \frac{\lambda_t^i}{\lambda_t^i - \lambda_t^j} = \frac{\beta(N-1)}{4N} \leq \frac{\beta}{4}$;
- (iii) (by convexity of V and $x \mapsto V(\sqrt{2x})$, and by Lemma 5.1) $V'(\lambda_t^i) \lambda_t^i \geq (1 - c_V)(V(|\lambda_t^i|) - V(x_V)) = (1 - c_V) \left(V\left(\sqrt{2 \cdot \frac{1}{2}(\lambda_t^i)^2}\right) - V(x_V) \right)$, whence $\frac{1}{N} \sum_{i=1}^N V'(\lambda_t^i) \lambda_t^i \geq (1 - c_V)(V(\sqrt{2\mathcal{V}_t^0}) - V(x_V))$.

All together, using the SDE comparison principle, we have $\mathcal{V}_t^0 \leq \tilde{\mathcal{V}}_t^0$, where $\tilde{\mathcal{V}}_t^0 \equiv \mathcal{V}_t^0$ and

$$d\tilde{\mathcal{V}}_t^0 = \frac{1}{N} \sqrt{2\tilde{\mathcal{V}}_t^0} dB_t + \frac{1}{2} \left(\frac{\beta}{2} + 1 \right) dt - (1 - c_V)(V(\sqrt{2\tilde{\mathcal{V}}_t^0}) - V(x_V)) dt. \quad (5.33)$$

Letting $R_t^0 := \sqrt{2\tilde{\mathcal{V}}_t^0}$, one finds

$$dR_t^0 = \frac{1}{N} dB_t + \frac{1}{2} \left(\frac{\beta}{2} + 1 \right) \frac{dt}{R_t^0} - (1 - c_V) \frac{V(R_t^0) - V(x_V)}{R_t^0} dt, \quad (5.34)$$

which is (5.17), up to the replacements $\sqrt{N} \rightarrow N$, $V'(x) \rightarrow (1 - c_V) \frac{V(x) - V(x_V)}{x}$. By principle we decide once and for all the all small constants c, c', \dots in the exponential rates have $1 - c_V$ in factor, and forget about it. Using the intermediate estimates (5.30) in the course of the proof of Lemma 5.5, we obtain the following large deviation estimates for $A^0 > \max(R_0^0, 3x_V)$: $\mathbb{P}[\sup_{t \in [0,1]} R_t^0 > A^0] \leq \frac{1}{(A^0)^2} e^{-cN^2(V(A^0) - V(x_V))}$ if $R_0^0 \leq \frac{2}{3}A^0$; and $\mathbb{P}[\sup_{t \in [0,1]} R_t^0 > A^0] \leq \frac{1}{(A^0 - R_0^0)^2} e^{-cN^2(A - R_0^0) \frac{V(A) - V(x_V)}{A}}$ if $R_0^0 > \frac{2}{3}A^0$. In the latter case, we use the regular variation hypothesis on V and the hypothesis $A \geq 3x_V$ to obtain: $(A - R_0^0) \frac{V(A) - V(x_V)}{A} \geq \frac{1}{2}(A - \max(R_0^0, x_V))V'(\frac{A}{2}) \geq c'(A - \max(R_0^0, x_V))V'(A) \geq V(A) - V(\max(R_0^0, x_V))$. Thus, in both cases, one concludes:

$$\mathbb{P}\left[\sup_{t \in [0,1]} R_t^0 > A^0\right] \leq \frac{1}{(A^0 - R_0^0)^2} \exp\left[-cN^2(V(A^0) - V(\max(R_0^0, x_V)))\right] \quad (5.35)$$

Remark: We do not really need the above computations in the case $R_0^0 > \frac{2}{3}A^0$, Namely, since

$$R_0^0 \leq \max_{i=1, \dots, N} |\lambda_0^i|, \quad (5.36)$$

Lemma 5.4 yields

$$\mathbb{P}[R_0^0 > \frac{2}{3}A^0] = O(e^{-cNV(\frac{2}{3}A^0)}) = O(e^{-c'NV(A^0)}), \quad (5.37)$$

a bound just as good as (5.35) *except* that the large deviation rate is now *linear* in N , instead of *quadratic*. Since in the next steps (see below) we eventually obtain linear large deviation rates in any case, we shall be content with using (5.35) only in the event $\{R_0^0 \leq \frac{2}{3}A^0\}$.

Now comes the key inductive argument. Because the law of $\{\lambda_t^i\}$ is symmetric under permutations of the eigenvalues, we may assume that the *absolute values* $(|\lambda_t^i|)_{i=1,\dots,N}$ are in decreasing order, namely,

$$|\lambda_t^N| \leq |\lambda_t^{N-1}| \leq \dots \leq |\lambda_t^2| \leq |\lambda_t^1|. \quad (5.38)$$

With this ordering convention,

$$(R_t^0)^2 \geq 2\mathcal{V}_t^0 \geq \frac{1}{N} \sum_{i=1}^{N/2} |\lambda_t^i|^2 \geq \frac{1}{2} |\lambda_t^{1+N/2}|^2, \quad (5.39)$$

an imperfect converse to (5.36). Combining (5.35) and (5.39), we get:

$$\mathbb{P}[\sup_{t \in [0,1]} |\lambda_t^{1+N/2}| > A^0 \sqrt{2}] \leq \frac{1}{(A^0 - R_0^0)^2} \exp[-cN^2(V(A^0) - V(\max(R_0^0, x_V)))] . \quad (5.40)$$

We describe the induction in very general terms to begin with. For convenience we assume for the proof that $N = 2^n$, and let $N_k := N/2^k = 2^{n-k}$ (for $N \neq 2^n$ the reader should just insert integer parts $\lfloor \cdot \rfloor$ at selected places in the text). For any increasing sequence $(A^k)_{k=0,\dots,n-1}$ such that

$$A^0 \geq 3x_V; \quad \forall k \geq 1, A^k - A^{k-1} \geq 2^{-k/2}/\beta, \quad (5.41)$$

we prove large deviation upper estimates for the probability of the event $\{\sup_{t \in [0,1]} |\lambda_t^{1+N_k}| > A_{k-1} \sqrt{2}\}$ on the event $\{\forall j < k, \sup_{t \in [0,1]} |\lambda_t^{1+N_j}| \leq A_{j-1} \sqrt{2}\}$. Conditions (5.41) ensure that the driving force on the quantities \mathcal{V}^k , $k = 0, \dots, n$ (see below) due to particles with smaller absolute value remains bounded; for that the condition is actually that the series in (5.51) should be bounded independently of n . In the best case $A^0 + \sum_{k=1}^{n-1} (A^k - A^{k-1}) \leq 3x_V + \frac{1}{\beta} \sum_{k=1}^{\infty} 2^{-k/2} = 3x_V + \frac{1}{\beta(\sqrt{2}-1)}$. So we actually get large deviation estimates for the probability that $\max_{i=1,\dots,N} \sup_{t \in [0,1]} |\lambda_t^j|$ be larger than the above number.

Let us start by describing step #1. Fix $A^0 \geq 3x_V$, and assume $|\lambda_0^{1+N_1}| = |\lambda_0^{1+N/2}| \leq A^0 + \frac{2}{3}(A^1 - A^0)$, an event of probability $1 - O(e^{-cNV(A^1)})$ by Lemma 5.4. By a slight abuse of notation, we still denote by $(\lambda_t^i)_{t \geq 0, i=1,\dots,N}$ the process *killed* when $|\lambda_t^{1+N_1}| > A^0$, and consider the step #1 analogue of the quantity \mathcal{V}_t^0 ,

$$\mathcal{V}_t^1 := \frac{1}{2N_1} \sum_{i \in I^1(t)} (|\lambda_t^i| - A^0)^2 = \frac{1}{2N_1} \sum_{i \in I^1(t)} (\lambda_t^i - \text{sgn}(\lambda_t^i) A^0)^2 \quad (5.42)$$

where $I^1(t) := \{i \mid |\lambda_t^i| > A^0\} \subset \{1, \dots, N_1\}$ (by assumption, $|\lambda_t^1| \geq \dots \geq |\lambda_t^N|$, so a.s. signs $\text{sgn}(\lambda_t^i)$, $i = 1, \dots, N-1$ are constant over time). Note that \mathcal{V}^1 is a continuous process. On every maximum time interval (t_i, t_f) on which $I^1(\cdot)$ is constant, we get by Itô's formula (compare with (5.32))

$$d\mathcal{V}_t^1 = \frac{1}{\sqrt{NN_1}} \sqrt{2\mathcal{V}_t^1} dB_t + \frac{1}{N_1} \left(\frac{N_1}{2N} + \frac{\beta}{2N} \sum_{i \neq j, i, j \in I^1} \frac{\lambda_t^i - \text{sgn}(\lambda_t^i) A^0}{\lambda_t^i - \lambda_t^j} + F_b(t) \right) dt - \frac{1}{N_1} \sum_{i \in I^1} V'(\lambda_t^i) (\lambda_t^i - \text{sgn}(\lambda_t^i) A^0) dt, \quad (5.43)$$

where $\frac{1}{N_1} F_b^1(t)$ is the force due to the background charges $\{\lambda_t^j\}_{j \notin I^1}$, explicitly,

$$F_b^1(t) := \frac{\beta}{2N} \sum_{i \neq j, i \in I^1, j \notin I^1} \frac{\lambda_t^i - \text{sgn}(\lambda_t^i) A^0}{\lambda_t^i - \lambda_t^j}. \quad (5.44)$$

Note the different N -prefactors with respect to (5.32). Then (emphasizing only the differences):

- (i) (by antisymmetry) let $\Lambda^1 := \{(i, j) \in I^1 \times I^1 \mid i \neq j, \text{sgn}(\lambda_t^i) \neq \text{sgn}(\lambda_t^j)\}$, then $\sum_{i \neq j, i, j \in I^1} \frac{\text{sgn}(\lambda_t^i) A^0}{\lambda_t^i - \lambda_t^j} = \frac{1}{2} \sum_{(i, j) \in \Lambda^1} \frac{A^0}{|\lambda_t^i| + |\lambda_t^j|} \leq \frac{1}{4} |\Lambda^1| \leq \frac{N^2}{8}$;
- (ii) (by convexity arguments) $V'(\lambda_t^i) (\lambda_t^i - \text{sgn}(\lambda_t^i) A^0) \geq (1 - c_V) V'(|\lambda_t^i|) (|\lambda_t^i| - A^0) \geq (1 - c_V) (V_{A_0}(|\lambda_t^i| - A^0) - V_{A_0}(0))$, where $V_{A_0}(\cdot) := V(A_0 + \cdot)$, whence $\frac{1}{N_1} \sum_{i \in I^1} V'(\lambda_t^i) (\lambda_t^i - \text{sgn}(\lambda_t^i) A^0) \geq V(A_0 + \sqrt{2\mathcal{V}_t^1}) - V(A^0)$;
- (iii) since by construction $0 \leq \frac{|\lambda_t^i| - A^0}{|\lambda_t^i - \lambda_t^j|} \leq 1$ for $i \in I^1, j \notin I^1$, the background contribution to the drift

$$|\frac{1}{N_1} F_b^1(t)| \leq \frac{\beta}{2NN_1} |I^1| (N - |I^1|) \leq \frac{\beta}{2NN_1} N_1^2 = \frac{\beta}{2} \frac{N_1}{N} \quad (5.45)$$

is of the same order as that due to the mutual interaction of the $\{\lambda^i\}_{i \in I^1}$, which is in absolute value $\leq \frac{1}{N_1} \frac{\beta}{2N} \frac{|I^1|^2}{2} \leq \frac{1}{N_1} \frac{\beta}{2N} \frac{N_1^2}{2} = \frac{\beta}{4} \frac{N_1}{N}$.

All together, we find $\mathcal{V}_t^1 \leq \tilde{\mathcal{V}}_t^1$, where

$$d\tilde{\mathcal{V}}_t^1 = \frac{1}{\sqrt{NN_1}} \sqrt{2\tilde{\mathcal{V}}_t^1} dB_t + \frac{1}{2} \frac{N_1}{N} \left(\frac{13\beta}{8} + 1 \right) dt - (V(\sqrt{2\tilde{\mathcal{V}}_t^1}) - V(0)) dt. \quad (5.46)$$

Letting $R_t^1 := \sqrt{2\tilde{\mathcal{V}}_t^1}$, one finds (5.17),

$$dR_t^1 = \frac{1}{\sqrt{NN_1}} dB_t + \frac{1}{2} \frac{N_1}{N} \left(\frac{13\beta}{8} + 1 \right) \frac{dt}{R_t^1} - \frac{V(R_t^1) - V(A^0)}{R_t^1} dt, \quad (5.47)$$

from which, for $A^1 > A^0$, using the obvious generalization of (5.39), $(R_t^1)^2 \geq \frac{1}{2} (|\lambda_t^{1+N_2}| - A^0)^2$ and once again (5.30),

$$\mathbb{P} \left[\left(|\lambda_0^{1+N_2}| \leq A^0 + \frac{2}{3}(A^1 - A^0) \right) \cap \left(\sup_{t \in [0,1]} R_t^1 > A^1 - A^0 \right) \right] \leq C \frac{e^{-cNN_1(V(A^1) - V(A^0))}}{(A^1 - A^0)^2}. \quad (5.48)$$

We may now describe step $\#k$ for arbitrary $k \in \{1, \dots, n\}$. Assume that the assumptions (5.41) on A^0, \dots, A^{k-1} are verified, and that $|\lambda_0^{1+N_j}| \leq A^{j-1} + \frac{2}{3}(A^j - A^{j-1})$, $j = 1, \dots, k$. Denote by $(\lambda_t^i)_{t \geq 0, i=1, \dots, N}$ the process killed when $|\lambda_t^{1+N_j}| > A^{j-1}$, $j = 1, \dots, k$, and let

$$\mathcal{V}_t^k := \frac{1}{2N_k} \sum_{i \in I^k(t)} (|\lambda_t^i| - A^{k-1})^2 \quad (5.49)$$

where $I^k(t) := \{i \mid |\lambda_t^i| > A^{k-1}\} \subset \{1, \dots, N_k\}$. Itô's formula (5.43) and (5.44) extends immediately to the case of \mathcal{V}_t^k , up to the modifications $N_1 \rightarrow N_k$, $I^1 \rightarrow I^k$, $F_b^1 \rightarrow F_b^k$ and $A^0 \rightarrow A^{k-1}$, and so do the ensuing estimates (i), (ii), and also the estimates (iii) for the contribution to the drift due to the mutual interaction of the $\{\lambda^i\}_{i \in I^k}$. On the other hand, the bound for the contribution due to the background force, although elementary, is more subtle; it highlights the coherence of the scheme. First, since $(A^j)_j$ is increasing, I^j is decreasing, so we may decompose the set of indices $I^0 \equiv \{1, \dots, N\}$ into the disjoint union $I^k \uplus (\uplus_{j=1, \dots, k} (I^{j-1} \setminus I^j))$. Using this decomposition, we find (bounding $\frac{1}{|\lambda_t^i - \lambda_t^p|}$, $i \in I^k, p \in \{1, \dots, N_j\} \setminus I^j$ by $\frac{1}{|\lambda_t^i| - A^{j-1}}$)

$$\begin{aligned} \left| \frac{1}{N_k} F_b^k(t) \right| &= \frac{\beta}{2N_k} \left| \sum_{i \neq j, i \in I^k, j \notin I^k} \frac{\lambda_t^i - \text{sgn}(\lambda_t^i) A^{k-1}}{\lambda_t^i - \lambda_t^j} \right| \\ &\leq \frac{\beta}{2N_k} \sum_{i \in I^k} (|\lambda_t^i| - A^{k-1}) \left\{ \frac{N_{k-1} - |I^k|}{|\lambda_t^i| - A^{k-1}} + \sum_{j=1}^{k-1} \frac{N_j}{|\lambda_t^i| - A^{j-1}} \right\} \\ &\leq \frac{\beta}{2} \frac{N_k}{N} + \frac{\beta}{2N_k} \sum_{i \in I^k} (|\lambda_t^i| - A^{k-1}) \sum_{j=1}^{k-1} \frac{N_j/N}{A^{k-1} - A^{j-1}}. \end{aligned} \quad (5.50)$$

Now, by (5.41)

$$\sum_{j=1}^{k-1} \frac{N_j/N}{A^{k-1} - A^{j-1}} \leq \sum_{j \geq 1} \frac{2^{-j}}{2^{-j/2}/\beta} \leq \frac{\beta}{\sqrt{2} - 1}. \quad (5.51)$$

The *first* term in the r.h.s. of (5.50) is of the same order as that due to the mutual interaction of the $\{\lambda^i\}_{i \in I^k}$. Leaving out the *second* term, we would get by the SDE comparison principle $\mathcal{V}_t^k \leq \tilde{\mathcal{V}}_t^k$, with \mathcal{V}_t^k and $R_t^k := \sqrt{2\mathcal{V}_t^k}$ satisfying resp. (5.46) and (5.47) with the obvious substitutions $N_1 \rightarrow N_k$, $A^0 \rightarrow A^{k-1}$. The *second* term, $\frac{\beta^2}{2(\sqrt{2}-1)} \frac{1}{N_k} \sum_{i \in I^k} (|\lambda_t^i| - A^{k-1})$ is not so good, but (by construction of x_V , see Lemma 5.1, point 1. (i)) it may be reabsorbed by Lemma 5.1 into the drift term due to the potential $-\frac{1}{N_k} \sum_{i \in I^k} V'(\lambda_t^i)(\lambda_t^i - \text{sgn}(\lambda_t^i) A^{k-1})$, yielding a total contribution

$\leq -\frac{1}{2} \frac{1}{N_k} \sum_{i \in I^k} V'(\lambda_t^k)(\lambda_t^i - \text{sgn}(\lambda_t^i) A^{k-1})$. Thus, finally,

$$\begin{aligned} & \mathbb{P} \left[\left(|\lambda_0^{1+N_{k+1}}| \leq A^{k-1} + \frac{2}{3}(A^k - A^{k-1}) \right) \cap \left(\sup_{t \in [0,1]} R_t^k > A^k - A^{k-1} \right) \right] \\ & \leq C \frac{e^{-cNN_k(V(A^k) - V(A^{k-1}))}}{(A^k - A^{k-1})^2} \end{aligned} \quad (5.52)$$

as for R^1 (see (5.47)).

The end of the argument for the proof of Theorem 5.1 goes as follows. Fix $R > 3x_V$. We want to bound $\mathbb{P}[\sup_{t \in [0,1]} |\lambda_t^1| > R\sqrt{2}]$. By construction $|\lambda_t^{N_1}| \leq |\lambda_t^{N_2}| \leq \dots \leq |\lambda_t^{N_n}| = |\lambda_t^1|$ for all t , so $\Lambda^1 \leq \Lambda^2 \leq \dots \leq \Lambda^n \equiv \sup_{t \in [0,1]} |\lambda_t^1|$, where

$$\Lambda^k := \sup_{t \in [0,1]} |\lambda_t^{N_k}|. \quad (5.53)$$

We cover the n -simplex $\{(\Lambda^1, \dots, \Lambda^n) \in \mathbb{R}^n \mid 0 \leq \Lambda^1 \leq \dots \leq \Lambda^n\}$ by isotropic n -cubes of the form $[p_1 2^{-n/2}, (p_1 + 1) 2^{-n/2}] \times \dots \times [p_n 2^{-n/2}, (p_n + 1) 2^{-n/2}]$, where $p_1 \leq \dots \leq p_n$ are integers, and let $(A^k)_{k=0, \dots, n-1}$ be the increasing sequence $A^k := p_{k+1} 2^{-n/2}$.

We first assume that condition (5.41) is verified, i.e. that the following event Ω holds,

$$\Omega : A \geq 3x_V; \quad A^k - A^{k-1} \geq 2^{-k/2}/\beta \quad (k \geq 1). \quad (5.54)$$

Being on Ω implies restricting the sum over the subset of cubes

$$\omega := \{p_1 \leq \dots \leq p_n \mid A^0 \geq 3x_V; \quad A^k - A^{k-1} \geq 2^{-k/2}/\beta \quad (k \geq 1)\}. \quad (5.55)$$

By construction, letting $\tilde{\Lambda}^k := \sup_{t \in [0,1]} |\lambda_t^{1+N_k}| \leq \Lambda^k$,

$$\begin{aligned} & \mathbb{P} \left[\Omega \cap (R < \sup_{t \in [0,1]} |\lambda_t^1| \leq 2R) \right] \leq \sum_{(p_1, \dots, p_n) \in \omega} \mathbf{1}_{R+O(2^{-n/2}) < A^{n-1} \leq 2R+O(2^{-n/2})} \\ & \quad \mathbb{P} \left[\Lambda^k = A^{k-1} \sqrt{2} + O(2^{-n/2}), k = 1, \dots, n \right], \end{aligned} \quad (5.56)$$

and

$$\mathbb{P} \left[\Lambda^k = A^{k-1} \sqrt{2} + O(2^{-n/2}), k = 1, \dots, n \right] \leq \min_{k=1, \dots, n} P_k, \quad (5.57)$$

where:

$$P_1 := \mathbb{P}[\Lambda^1 \geq A^0 \sqrt{2} - O(2^{-n/2})] \quad (5.58)$$

and, for $k = 2, \dots, n$,

$$P_k := \mathbb{P} \left[\left(\forall j = 1, \dots, k-1, \tilde{\Lambda}^j \leq A^{j-1} \sqrt{2} + O(2^{-n/2}) \right) \cap \left(\Lambda^k \geq A^{k-1} \sqrt{2} - O(2^{-n/2}) \right) \right]. \quad (5.59)$$

It would be more pleasant to have $\prod_{k=1}^n P_k$ instead of $\min_k P_k$ in the r.-h.s. of (5.57), which would give a deviation rate *linear in N* , but we clearly lack independence. Proceeding further, we have:

$$\begin{aligned}
P_1 &\leq \mathbb{P}[|\lambda_0^{N_1}| > \frac{2}{3}(A^0\sqrt{2} - O(2^{-n/2}))] \\
&\quad + \mathbb{P}\left[\left(|\lambda_0^{N_1}| \leq \frac{2}{3}(A^0\sqrt{2} - O(2^{-n/2}))\right) \cap \left(\Lambda^1 \geq A^0\sqrt{2} - O(2^{-n/2})\right)\right] \\
&= O\left(\frac{e^{-cN(V(A^0)-V(x_V))}}{(A^0)^2}\right)
\end{aligned} \tag{5.60}$$

by (5.35) and (5.37).

Similarly, for $k = 1, \dots, n-1$,

$$\begin{aligned}
P_{k+1} &\leq \mathbb{P}[|\lambda_0^{N_{k+1}}| > \sqrt{2}(A^{k-1} + \frac{2}{3}(A^k - A^{k-1})) - O(2^{-n/2})] \\
&\quad + \mathbb{P}\left[\left(|\lambda_0^{N_{k+1}}| \leq \sqrt{2}(A^{k-1} + \frac{2}{3}(A^k - A^{k-1})) - O(2^{-n/2})\right) \cap \left(\Lambda^{k+1} \geq A^k\sqrt{2} - O(2^{-n/2})\right)\right] \\
&= O\left(\frac{e^{-cN(V(A^k)-V(A^{k-1}))}}{(A^k - A^{k-1})^2}\right)
\end{aligned} \tag{5.61}$$

Hence (using (5.41))

$$\min_{k=1, \dots, n} P_k \leq (P_1 \cdots P_n)^{1/n} = O\left(e^{-c\frac{N}{n}(V(A^{n-1})-V(x_V))} \prod_{k=1}^n (\beta^2 2^k)\right). \tag{5.62}$$

The sum in (5.56) is thus less than a constant times $O(e^{-c\frac{N}{n}(V(R)-V(x_V))})$, times $C^n 2^{O(n^2)}$, times the total number of cubes in ω such that $A^{n-1} \leq 2R + O(2^{-n/2})$, which is $\leq C^n \prod_{k=1}^n (2^{n/2}R) \leq (CR)^n 2^{O(n^2)}$, a volume factor which is negligible with respect to $e^{c\frac{N}{n}(V(R/\sqrt{2})-V(x_V))}$ for $R \geq 3x_V$ and $N = 2^n$ large enough.

Now, if condition (5.41) is not satisfied, then we regroup dyadic packets of eigenvalues. If e.g. $A^0 \geq 3x_V$ and $A^j - A^{j-1} \geq 2^{-k/2}/\beta$ for all $j < k$, but $A^k - A^{k-1} \geq 2^{-k/2}/\beta$ then we look for the smallest integer $k' \geq k$ such that $A^{k'} - A^{k-1} \geq 2^{-k/2}/\beta$, and iterate the procedure starting from index k' , thus generating a subset of indices $k_0 < k_1 < k_2 < \dots$ included in $\{0, \dots, n-1\}$. Then we bound (5.57) by $\min_{k \in \{k_0, k_1, k_2, \dots\}} P_k$, obtaining the same estimates, multiplied by $O(2^n)$ (counting the number of such subsets of indices), which may again be reabsorbed into the decreasing exponential.

Finally, using

$$\mathbb{P}\left[\sup_{t \in [0,1]} |\lambda_t^1| > R\right] = \sum_{j=0}^{+\infty} \mathbb{P}[2^j R < \sup_{t \in [0,1]} |\lambda_t^1| \leq 2^{j+1} R] \tag{5.63}$$

yields Theorem 5.1. \square

6 Appendix. Generalized transport operators

Many operators in this article are of the following type,

$$\mathcal{H}_t f(x) = \sum_k v_k(t, x) \partial_{x_k} f + \tau(t, x) \quad (6.1)$$

with $f : \Omega \rightarrow \mathbb{R}$, where Ω is a domain in \mathbb{R}^d (in practice, we need only consider $\Omega = \Pi^\pm$), and $\mathbf{v}(t, \cdot)$ a vector field, resp. $\tau(t, \cdot)$ a function, on Ω . Let us call such operators *generalized transport operators*.

It is well-known how to solve PDEs generated by generalized transport operators, i.e. of the type

$$\frac{\partial f_t}{\partial t}(x) = \mathcal{H}_t f_t(x) \quad (6.2)$$

with terminal condition $f_T \equiv f$. Namely, let $y_t \equiv \Phi_t^T(y)$ (called: *characteristics of (6.2)*) be the solution of the ode $\frac{dy_t}{dt} = \mathbf{v}(t, y_t)$ with terminal condition $y_T = y$. One checks immediately that

$$f_t(y) = c_t f(\Phi_t^T(y)), \quad c_t := \exp \left(- \int_t^T \tau(y_s) ds \right) \quad (6.3)$$

is a solution. In particular, $\text{supp}(f_t), t \leq T$ is the inverse image of $\text{supp}(f)$ by Φ_t^T ; so, if $\mathbf{v}|_{\partial\Omega}$ is *inward on the boundary of some domain Ω containing the support of f_T* , then $\text{supp}(f_t) \subset \Omega$ for all $t \leq T$. In the article we actually refer either to the basis trajectory $y. = a. + ib.$ or to the "extended" trajectory $(a. + ib., c.)$ as characteristics.

The *Jacobian* of the ode, $J_t := \frac{dy_t}{dy}$, solves the linearized ode $\frac{dJ_t}{dt} = \nabla \mathbf{v}(t, y_t) J_t$ with terminal condition $J_T = \text{Id}$. In particular (letting $|\cdot|$ denote the determinant), $\frac{d}{dt}|J_t|_{t=T} = \text{Tr}(\nabla \mathbf{v}(T, y)) = \nabla \cdot \mathbf{v}(T, y)$. The time-variation of the L^1 -norm of f_t is

$$\begin{aligned} \frac{d}{dt}|J_t|_{t=T} \int dy |f_t(y)| &= \int dy \left(\text{Re} \frac{d}{dt} c_t(y) \right) |f(y)| - \int dy \left(\frac{d}{dt}|J_t|_{t=T} \right) |f(y)| \\ &= \int dy (\text{Re} \tau(T, y) - \text{Tr}(\nabla \mathbf{v}(T, y))) |f(y)|; \end{aligned} \quad (6.4)$$

it vanishes when $\text{Re } c = \text{Tr}(\nabla \mathbf{v})$, in particular when

$$\mathcal{H}_t = \left(\sum_k v_k(t, x) \partial_{x_k} \right)^\dagger = - \sum_k v_k(t, x) \partial_{x_k} - \nabla \cdot \mathbf{v}(t, x) \quad (6.5)$$

is in *divergence form*, i.e. is the *adjoint* of a transport operator. Thus \mathcal{H}_t is the generator of a strongly continuous semi-group of contractions of $L^1(\mathbb{R})$, see e.g. [25], chapter 1. The latter observation extends to the case when $\mathcal{H}_t = (\sum_k v_k(t, x) \partial_{x_k})^\dagger - \tau(t, x)$ with $\tau(t, \cdot) \leq 0$, in the sense that $\int dy |f_t(y)| \leq \int dy |f_T(y)|$ for $0 \leq t \leq T$.

7 Appendix. Stieltjes transforms

We collect in this section some definitions and elementary properties concerning Stieltjes transforms. We make use of the Fourier transform normalized as follows,

$$\mathcal{F}(f)(s) = \int_{-\infty}^{+\infty} f(x) e^{-ixs} dx \quad (7.1)$$

with inverse $\mathcal{F}^{-1}(g)(s) = \frac{1}{2\pi} \int g(s) e^{ixs} ds$.

Let, for $z = a + ib \in \mathbb{C} \setminus \mathbb{R}$

$$f_z(x) = \frac{1}{x - z}, \quad x \in \mathbb{R}. \quad (7.2)$$

For fixed $b \neq 0$, $f_z(x)$ may be seen as a convolution kernel K_b ,

$$K_b(x - a) = \frac{1}{(x - a) - ib}. \quad (7.3)$$

Note that

$$\operatorname{Im} (f_z)(x) = \frac{b}{|x - z|^2} = \frac{b}{(x - a)^2 + b^2}, \quad \operatorname{Re} (f_z)(x) = \frac{x - a}{(x - a)^2 + b^2}. \quad (7.4)$$

In particular,

$$\operatorname{Im} (f_z)(x) \geq 0 \quad (b > 0). \quad (7.5)$$

Many estimates are based on the simple remark that

$$\int \frac{b}{(x - a)^2 + b^2} dx = \pi \quad (b > 0) \quad (7.6)$$

is a constant. The Plemelj formula,

$$\frac{1}{x - i0} = p.v. \left(\frac{1}{x} \right) + i\pi\delta_0 \quad (7.7)$$

implies the following boundary value equations for K_b ,

$$\lim_{b \rightarrow 0^+} \int dy K_b(x - y) \phi(y) - \lim_{b \rightarrow 0^-} \int dy K_b(x - y) \phi(y) = 2i\pi\phi(x) \quad (7.8)$$

$$\lim_{b \rightarrow 0^+} \int dy K_b(x - y) \phi(y) + \lim_{b \rightarrow 0^-} \int dy K_b(x - y) \phi(y) = 2 p.v. \int \frac{dy}{x - y} \phi(y) \quad (7.9)$$

Then:

$$\mathcal{F} f_z(s) = 2i\pi e^{-b|s| - ias} \mathbf{1}_{s < 0} \quad (b > 0), \quad -2i\pi e^{b|s| - ias} \mathbf{1}_{s > 0} \quad (b < 0) \quad (7.10)$$

hence (for $b > 0$)

$$\mathcal{F}(\operatorname{Im} (f_z))(s) = \pi e^{-b|s| - ias}, \quad \mathcal{F}(\operatorname{Re} (f_z)) = -i\pi \operatorname{sgn}(s) e^{-b|s| - ias}. \quad (7.11)$$

Properties of the Stieltjes transform of ρ_t .

Let $M_t(z) := \langle X_t, f_z \rangle$ ($b := \text{Im } z > 0$). Then:

$$\text{Im } (M_t(z)) = \langle X_t, \frac{b}{(x-a)^2 + b^2} \rangle \in [0, \pi \|\rho_t\|_\infty]; \quad (7.12)$$

$$|M_t(z)| = |\langle X_t, \frac{1}{(x-a) - ib} \rangle| \leq 1/b; \quad (7.13)$$

$$|M'_t(z)| = |\langle X_t, \frac{1}{((x-a) - ib)^2} \rangle| \leq \frac{1}{b} \langle X_t, \frac{b}{(x-a)^2 + b^2} \rangle = \frac{1}{b} \text{Im } (M_t(z)). \quad (7.14)$$

When $|a| \gg R$, we get much better estimates, e.g.

$$|M_t(z)| \leq 2/|a|, \quad |a| \geq 2R. \quad (7.15)$$

On the other hand, if $b \rightarrow 0$ and $a \in \text{supp}(X_t)$, then one expects $\text{Re } (M_t(z))$ to diverge logarithmically in general. In particular,

$$|\text{Re } M_t(z)| \leq C \|\rho_t\|_\infty \ln(R/b) \quad (b \leq \frac{R}{2}, |a| \leq 2R), \quad (7.16)$$

compare with (7.12). In any case, we do *not* use estimates (7.12) and (7.16) in the text because they do *not* hold for $M_t^N(z) := \langle X_t^N, f_z \rangle$ since X_t^N has no density. On the other hand, (7.13), (7.14), (7.15) also hold for M_t^N .

Some distributions.

Let $\phi \in C_c^\infty$ be a smooth function supported on $[-r, r]$, and $b > 0$. Let

$$\langle f_{ib}, \phi \rangle := \int dy \frac{\phi(y)}{y - ib} \quad (7.17)$$

Then

$$\langle f_{ib}, \phi \rangle = \phi(0) \int_{-r}^r \frac{dy}{y - ib} + i \int_{-r}^r dy (\phi(y) - \phi(0)) \frac{b}{y^2 + |b_T|^2} + \int_{-r}^r dy \frac{y(\phi(y) - \phi(0))}{y^2 + b^2} \quad (7.18)$$

is $O(\|\phi\|_\infty + r\|\phi'\|_\infty)$ since: $|\int_{-r}^r \frac{dy}{y - ib}| \leq \int dy \frac{b}{y^2 + b^2} = O(1)$, and $|\frac{y(\phi(y) - \phi(0))}{y^2 + b^2}| \leq \|\phi'\|_\infty$. Hence (by differentiating), $y \mapsto (y - ib_T)^{-n}$ ($n \geq 1$) is a distribution of order n , namely,

$$\left| \int dy \frac{\phi(y)}{(y - ib)^n} \right| = O(\|\phi^{(n-1)}\|_\infty + r\|\phi^{(n)}\|_\infty). \quad (7.19)$$

The $(\frac{1}{x})$ -kernel and its family.

It is known that

$$\mathcal{F}^{-1}(s \mapsto \text{sgn}(s) \mathcal{F}f(s))(x) = iHf(x) := \frac{i}{\pi} p.v. \int_{-\infty}^{+\infty} \frac{1}{x-y} f(y) dy \quad (7.20)$$

defined for a compactly supported $f \in C^1$ either as $\frac{i}{\pi} \lim_{\varepsilon \rightarrow 0^+} \int_{|x-y|>\varepsilon} \frac{1}{x-y} f(y) dy$ or as $\frac{i}{\pi} \int \frac{f(y)-f(x)}{x-y} dy$, from which by differentiating

$$\mathcal{F}^{-1}(s \mapsto |s| \mathcal{F}f(s))(x) = -\frac{1}{\pi} p.v. \int_{-\infty}^{+\infty} \frac{1}{(x-y)^2} f(y) dy \quad (7.21)$$

For a function f supported on $[-R, R]$, we have the following bounds:

$$\begin{aligned} \left| p.v. \int \frac{1}{x-y} f(y) dy \right| &= \mathbf{1}_{|x| \leq 2R} \left| \int_{x-3R}^{x+3R} \frac{f(y) - f(x)}{x-y} dy \right| + \mathbf{1}_{|x| > 2R} \left| \int_{-R}^R \frac{f(y)}{x-y} dy \right| \\ &= \mathbf{1}_{|x| \leq 2R} O(R \|f'\|_{\infty}) + \mathbf{1}_{|x| > 2R} O(\|f\|_{\infty} R/|x|), \end{aligned} \quad (7.22)$$

and similarly

$$\begin{aligned} \left| p.v. \int \frac{1}{(x-y)^2} f(y) dy \right| &= \mathbf{1}_{|x| \leq 2R} \left| \int_{x-3R}^{x+3R} \frac{f'(y) - f'(x)}{x-y} dy \right| + \mathbf{1}_{|x| > 2R} \left| \int_{-R}^R \frac{f(y)}{(x-y)^2} dy \right| \\ &= \mathbf{1}_{|x| \leq 2R} O(R \|f''\|_{\infty}) + \mathbf{1}_{|x| > 2R} O(\|f\|_{\infty} R/x^2). \end{aligned} \quad (7.23)$$

Acknowledgements. The author had the opportunity and the pleasure to discuss this project at several occasions and places with N. Simm (Univ. of Warwick), to whom he therefore wishes to express his gratitude.

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